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Hydrodynamic Limit for an Arc Discharge at Atmospheric Pressure

Isabelle Choquet[1] **and Brigitte Lucquin-Desreux**[2]

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In this paper we study a partially ionized plasma that corresponds to an arc discharge at atmospheric pressure. We derive an inviscid hydrodynamic/diffusion limit, characterized by two temperatures, from a system of Boltzmann type transport equations modelling that plasma problem. The original property of this system is that impact ionization is a leading order collisional process. As a consequence, the density of electrons is given in terms of the density of the other species (and its temperature) via a Saha law.

KEY WORDS: Arc discharge; gas mixture; disparate masses; impact ionization; inelastic collisions; fluid limit; Hilbert expansion; Saha plasma.

1. INTRODUCTION

The scientific study of electric discharges started during the 18th century with the experimental observation of sparks produced by electrostatic generators and thunderstorms lightnings. The modelling of a discharge (a glow discharge in uniform field) was first investigated in the early 1900s by Townsend, also Stark. Nowadays there exists a large variety of electric discharges used for many industrial applications. Here, we will focus on a specific one: the arc discharge at atmospheric pressure. It can be produced with moderate voltage of some Volts, or characterized by a low current of the order of one Ampère. Its fluid dynamic and thermal effects are

[1]Department of Technology, Trollhättan/Uddevalla University, Box 957, SE-461 39 Trollhättan; e-mail: isabelle.choquet@htu.se
[2]Laboratoire d'Analyse Numérique, Université Pierre et Marie Curie, 16 rue Clisson, F-75013 Paris.

dominant, allowing many applications based on energy transfer. Among them are transferred arcs used in metallurgy to cut, melt or weld; their power lies within the range 10^2 to 10^4 W. Another important application field concerns plasma torches used to spray protective coatings (for a better resistance to wear, corrosion, oxidation, thermal fluxes, also for electric or electronic purposes and bio-compatibility) or vitrify toxic wastes for instance; their power can go from 10^2 up to 10^5 W.

The modelling of arc discharges covers a wide range of complex physical phenomena (electric, magnetic, thermal, chemical, and fluid dynamic effects) occurring in different regions of the arc and, moreover, involving different scales. An arc is usually modelled dividing it into a thermal plasma column (which represents the main body) surrounded by electrode layers. From the cathode surface, the boundary layer includes first a very thin space charge zone of the order of one electron mean free path (or $\approx 10^{-6}$ m for the Argon arc of 200 A and 1700 W studied in ref. 1 for instance) where the transition between metallic and gaseous conduction is done. The number of collisions in this sheath is negligible; there is thus no local thermodynamic equilibrium, no ionization and no recombination. There, ions are freely accelerated towards the cathode, and their collision with the metallic surface results in thermionic emission of electrons. A considerable current is thus built up by these ions, supplemented by thermionic and secondary electrons as well as electrons counter-diffusing against the electric field of the sheath. During their acceleration across the sheath, towards the pre-sheath, the emitted electrons can acquire a kinetic energy larger than the ionization threshold of the plasma gas.

In the next sub-layer of the boundary, or pre-sheath, ionization dominates over recombination. This ionization zone does the link with the plasma column. Its thickness is of the order of the recombination length (or $\approx 10^{-4}$ m for the previous example,[(1)]). For an arc discharge, e.g. a weakly ionized plasma, the recombination length is much larger than the mean free path. The ionization zone is thus in local thermal equilibrium. But this equilibrium is partial: the electron temperature is almost twice larger than the heavy particle temperature. As the Debye length ($\approx 10^{-5}$ m concerning the example of ref. 1) is also less than the ionization mean free path, the pre-sheath is quasi-neutral.

The anode layer can also be divided into a sheath and pre-sheath structure, although the distribution of the electric potential differs. The sheath is also characterized by a deviation from local thermodynamic equilibrium. There the temperature of heavy particles accommodates with the anode surface temperature, while the electron temperature remains large enough to ensure electrical conductivity. The modelling of this part is generally developed doing simplifying assumptions suited to the numerical simulation of specific applications. For instance stationary with cylindrical symmetry for tungsten inert gas (TIG) welding arcs as in ref. 2 or non-stationary and 3-dimensional for plasma spraying arcs as in ref. 3. In that later case the arc attachment may move under the combined influence of the gas flow and Lorentz forces, inducing variations of the arc length and thus the arc voltage.

The plasma column is the most extended part of the arc, of the order of 10^{-2} m. It is thus in local thermal equilibrium (generally partial) and free of space charge, i.e. quasi-neutral. In this zone, both ionization and recombination occur.

As the treatment of the boundary conditions is a critical issue for numerical applications, most of the recent developments on the modelling of arc discharges are focussed on improved descriptions of sheath and pre-sheath, as in refs. 4 and 5 for instance. These authors start from fundamental physical principles to set-up consistent models for the various parts of the arc discharge, mentioned above. In refs. 3 and 6 the plasma column is modelled using the Navier–Stokes equations for one fluid in the presence of electro-magnetic forces. On the other hand, in ref. 5 a two temperature mixture is considered, but both temperatures and pressure are supposed to be given and the Saha law (as a function of the electron temperature) is used jointly with the ideal gas law to calculate the particle densities. Concerning the pre-sheath, refs. 3 and 6 do not make any distinction with the plasma column. Most of the recent models specifically developed for the ionization pre-sheath are directly derived from fluid equations rather than from the kinetic scale. They consider, as in ref. 5, a mixture of ions, electrons and neutrals characterized by two temperatures (one for electrons, one for heavy particles) and ionization reactions. There exist a model derived from kinetic theory in ref. 1 to establish mass and energy conservation equations assuming a steady flow, charge neutrality and equilibrium composition. It is based on the formulation of the electron diffusion flux proposed by Devoto.[7] This formulation is derived assuming that the electron Boltzmann equation can be decoupled from the heavy species Boltzmann equations and using a modified version[8] of the Chapman–Enskog method.[9]

In this study, we start from the kinetic scale to derive a macroscopic hydrodynamic/diffusion limit suited to the modelling of both the plasma column and the pre-sheaths (or ionization layers) of an atmospheric arc discharge. We will see in the sequel that the coupling between electrons and heavy species plays a major role. The framework is as follows: we consider a weakly ionized plasma, assume partial local thermal equilibrium and quasi-neutrality. We also account for impact ionization and recombination, and neglect radiative ionization and recombination (that get significant for high pressure discharges). We thus investigate a partially ionized plasma whose electrons, ions and neutral molecules are subject to elastic binary collisions as well as impact ionization and its reverse recombination reaction. The activation energy Δ of ionization reactions is supposed to be constant and given by the impacting electron. We also assume that the ionization level lies within the range 10^{-3} to 10^{-1}, which corresponds to an arc discharge problem.

Let us recall that the derivation of hydrodynamic/diffusion limits for a binary plasma gas mixture can be found in ref. 10 for instance. The ternary gas mixture corresponding to a very weakly ionized plasma, such as a glow discharge where ionization occurs very seldom, is studied in ref. 11. A problem with dominant impact ionization and its reverse recombination is investigated in refs. 12, 13 within the frame of semiconductors. An important difference compared to the present study is due to the masses of the charged particles of opposite sign: they have the same order of magnitude for semiconductors while they differ by orders of magnitude for plasma applications.

We start this study from a system of Boltzmann type transport equations governing the distribution functions of electrons, ions and neutral molecules. This system, presented in Section 2, is coupled through collision operators that involve three collisional processes: (i) elastic binary collisions where at least one particle is neutral (Boltzmann), (ii) elastic binary collisions between charged particles (Fokker–Planck), and (iii) inelastic collisions with impact ionization and its reverse recombination. This system is scaled in Section 3, based on its two small parameters. The first parameter ϵ measures the relative smallness of the electron mass with respect to the neutral particles. The second parameter δ measures the ionization level of the plasma. For an arc discharge we have $\delta \approx \epsilon$. The main consequence of this scaling is that impact ionization gets a leading order collisional process. Section 4 is devoted to some preparatory results concerning the leading order collision operators. A macroscopic limit is then presented in Section 5, while the proofs are detailed in Section 6. The two main results are contained in Theorem 5.1, where the equilibrium states and the Saha law are derived, and in Theorem 5.6 for our hydrodynamic model with two temperatures. We observe that it presents some differences compared to the models mentioned above. Part of these differences already appear with the inviscid fluid limit we propose to derive in this paper. The viscous case is the object of a forthcoming study.[14]

2. A SYSTEM OF KINETIC EQUATIONS MODELLING AN ARC DISCHARGE

We study a partially ionized gas whose electrons, ions and neutral molecules are subject to elastic binary collisions as well as ionization and recombination reactions. To that purpose, we assume that:

Assumption 2.1. The external forces applied to the particles do not depend on the velocity variable, excluding thus magnetized plasmas.

Assumption 2.2. The interaction potentials associated with the non-reactive collisions only depend on the distance between the particles.

Assumption 2.3. Radiative ionization and recombination are negligible, excluding thus very high pressure arc discharges.

Assumption 2.4. The activation energy of impact ionization reactions is given by the electron, and not by a heavy particle.

Assumption 2.5. The charge level of ions is of order one.

Notice that relaxing condition 2.1, we will make a more general case in the last remark of Section 5.

To avoid as much as possible confused notations, we restrict the presentation of this study to a single neutral species and the related single charged ion species. The extension of the forthcoming results to several molecular species characterized by masses of similar order of magnitude, and to ions of charge $Z < 10$ is indeed straightforward.

We thus investigate the following system of kinetic equations:

$$\begin{aligned}
&\partial_t f^e + v_e \cdot \nabla_x f^e + \frac{F_e}{m_e} \cdot \nabla_{v_e} f^e = (\partial_t f^e)_c, \\
&\partial_t f^i + v_i \cdot \nabla_x f^i + \frac{F_i}{m_i} \cdot \nabla_{v_i} f^i = (\partial_t f^i)_c, \\
&\partial_t f^n + v_n \cdot \nabla_x f^n + \frac{F_n}{m_n} \cdot \nabla_{v_n} f^n = (\partial_t f^n)_c.
\end{aligned} \tag{1}$$

The indexes e, i and n denote quantities associated with electrons, ions and neutral particles, respectively. The distribution functions $f^\alpha = f^\alpha(t, x, v_\alpha)$, where $\alpha = e, i, n$, depend on time $t \geqslant 0$, space $x \in I\!R^3$ and velocity $v_\alpha \in I\!R^3$. The force terms F_α represent the external forces acting on the particle α of mass m_α; it satisfies the Assumption 2.1.

The system of equations (1) is coupled through the source terms $(\partial_t f^\alpha)_c$ which modelize the collisions between particles. These operators involve three collisional processes detailed below: (i) elastic binary collisions where at least one particle is neutral, (ii) elastic binary collisions between charged particles, and (iii) inelastic collisions with ionization and its reverse recombination. Thus

$$\left(\partial_t f^\alpha\right)_c = Q^{\alpha\alpha}(f^\alpha, f^\alpha) + Q^{\alpha\beta}(f^\alpha, f^\beta) + Q^{\alpha\gamma}(f^\alpha, f^\gamma) + Q^{\alpha,ir}(f^\alpha, f^\beta, f^\gamma), \tag{2}$$

where the superscript *ir* stands for ionization-recombination, and $\alpha, \beta, \gamma = e, i, n$ with $\alpha \neq \beta \neq \gamma \neq \alpha$.

Let us first consider binary elastic collisions between the two particles α and β. When one of these particles (or both) is neutral, the binary collisions are described by Boltzmann operators of the form:

$$Q^{\alpha\beta}(f^\alpha, f^\beta)(v_\alpha) = \int_{I\!R^3 \times S_+^2} \sigma_{\alpha\beta}^{\mathcal{B}} \; |v_\alpha - v_\beta^\star| \Big(f^{\alpha\prime} f_\star^{\beta\prime} - f^\alpha f_\star^\beta \Big) \; dv_\beta^\star d\Omega, \tag{3}$$

where $\alpha = n$ and $\beta = e, i, n$ or $\alpha = e, i, n$ and $\beta = n$. In this expression, v_α [resp. $v_\beta^\star$] is the velocity of particle α [resp. β] before collision, and f^α [resp. $f_\star^\beta$] denotes: $f^\alpha = f^\alpha(t, x, v_\alpha)$ [resp. $f_\star^\beta = f^\beta(t, x, v_\beta^\star)$]. The post-collisional velocities $v_\alpha{}'$ and $v_\beta^{\star\prime}$ are defined from the pre-collisional velocities v_α and $v_\beta^\star$ by

$$v_\alpha{}' = v_\alpha - 2\frac{\mu_{\alpha\beta}}{m_\alpha}\Big((v_\alpha - v_\beta^\star)\cdot\Omega\Big)\Omega \quad \text{and} \quad v_\beta^{\star\prime} = v_\beta^\star + 2\frac{\mu_{\alpha\beta}}{m_\beta}\Big((v_\alpha - v_\beta^\star)\cdot\Omega\Big)\Omega, \tag{4}$$

where $\mu_{\alpha\beta} = m_\alpha m_\beta/(m_\alpha + m_\beta)$ is the reduced mass, and $\Omega \in S_+^2$ denotes a unit vector of part of the unit sphere S^2 of $I\!R^3$ defined by $S_+^2 := \Big\{\Omega \in S^2; (v_\alpha - v_\beta^\star)\cdot\Omega > 0\Big\}$. The notations $f^{\alpha\prime}$ and $f_\star^{\beta\prime}$ stand for $f^\alpha(t, x, v_\alpha{}')$ and $f^\beta(t, x, v_\beta^{\star\prime})$, respectively.

From Assumption 2.2, the scattering cross section $\sigma_{\alpha\beta}^{\mathcal{B}}$ is a function of two variables:

$$\sigma_{\alpha\beta}^{\mathcal{B}} = \sigma_{\alpha\beta}^{\mathcal{B}}(\mathcal{E}, \chi),$$

where $\mathcal{E} = \mu_{\alpha\beta}|v_\alpha - v_\beta^\star|^2$ is the reduced kinetic energy and χ denotes the angle $\left(\frac{v_\alpha - v_\beta^\star}{|v_\alpha - v_\beta^\star|}, \Omega\right)$. While the former belongs to $I\!R_+$, the latter lies within the range [0, 1].

Elastic collisions between two charged particles α and β are modelled by Fokker–Planck–Landau operators:

$$Q^{\alpha\beta}(f^\alpha, f^\beta)(v_\alpha) = \frac{\mu_{\alpha\beta}^2}{m_\alpha} \nabla_{v_\alpha} \cdot \Big[\int_{IR^3} \sigma_{\alpha\beta}^{\mathcal{F}} \, |v_\alpha - v_\beta^\star|^3 \, S(v_\alpha - v_\beta^\star) \times \Big(\frac{1}{m_\alpha} \nabla_{v_\alpha} f^\alpha f_\star^\beta - \frac{1}{m_\beta} \nabla_{v_\beta^\star} f_\star^\beta f^\alpha \Big) \, dv_\beta^\star \Big],$$

where $\alpha, \beta = e, i$ and $\nabla_{v_\alpha} f^\alpha = (\nabla f^\alpha)(v_\alpha)$, while $S(w)$ denotes the matrix $S(w) = \mathbf{Id} - \frac{w \otimes w}{|w|^2}$, **Id** being the identity matrix. Here, due to Assumption 2.2, the scattering cross section for grazing collisions $\sigma_{\alpha\beta}^{\mathcal{F}}$ only depends on the reduced kinetic energy:

$$\sigma_{\alpha\beta}^{\mathcal{F}} = \sigma_{\alpha\beta}^{\mathcal{F}}(\mathcal{E}) \, .$$

From Assumption 2.3, radiative ionization and recombination are supposed to be negligible. The ionization process we consider is thus impact ionization. Its mechanism can be schematized by the following direct and reverse reactions:

$$e + A \overset{\sigma^d}{\longrightarrow} e + e + A^+ \quad \text{and} \quad e + A \overset{\sigma^r}{\longleftarrow} e + e + A^+, \tag{5}$$

where e represents an electron, A^+ a single charged ion, and A the related neutral atom. σ^d and σ^r stand for the direct and reverse reaction cross sections. They are supposed to be positive. Applying the principle of detailed balance, we assume in the sequel that these cross sections are linked through

$$\sigma^d = \mathcal{F}_0 \, \sigma^r, \tag{6}$$

where $\mathcal{F}_0$ is a positive constant, which represents the efficiency of the dissociation with respect to the recombination. The ionization-recombination operators are then given by

$$Q^{e,ir}(f^e, f^i, f^n)(v_e) = \int_{IR^{12}} \sigma^r \delta_v \delta_{\mathcal{E}} \Big(f^{e\prime} f_\star^e f^i - \mathcal{F}_0 f^e f^n \Big) \, dv_e{}' \, dv_e^\star \, dv_i \, dv_n + 2 \int_{IR^{12}} \sigma^{r\prime} \delta_{v'} \delta_{\mathcal{E}'} \Big(\mathcal{F}_0 f^{e\prime} f^n - f^e f_\star^e f^i \Big) dv_e{}' \, dv_e^\star \, dv_i \, dv_n, \tag{7a}$$

$$Q^{i,ir}(f^e, f^i, f^n)(v_i) = \int_{IR^{12}} \sigma^r \delta_v \delta_{\mathcal{E}} \left(\mathcal{F}_0 f^e f^n - f^{e\prime} f^e_\star f^i\right) dv_e \, dv_e{}' \, dv_e^\star \, dv_n, \tag{7b}$$

$$Q^{n,ir}(f^e, f^i, f^n)(v_n) = \int_{IR^{12}} \sigma^r \delta_v \delta_{\mathcal{E}} \left(f^{e\prime} f^e_\star f^i - \mathcal{F}_0 f^e f^n\right) dv_e \, dv_e{}' \, dv_e^\star \, dv_i. \tag{7c}$$

Taking into account Assumption 2.4, the reverse reaction cross section writes

$$\sigma^r = \sigma^r(v_e{}', v_e^\star, v_i; v_e, v_n) = \sigma^r(v_e{}', v_e^\star; v_e), \tag{8}$$

and $\sigma^{r\prime} = \sigma^r(v_e, v_e^\star; v_e{}')$. The notations $\delta_{\mathcal{E}}$ and δ_v hold for the energy and momentum conservation during the ionization-recombination process; more precisely, we have:

$$\begin{aligned} \delta_{\mathcal{E}} &= \delta\Big(m_e|v_e|^2 + m_n|v_n|^2 - [m_e(|v_e{}'|^2 + |v_e^\star|^2) + m_i|v_i|^2 + 2\Delta]\Big), \\ \delta_v &= \delta\Big(m_e v_e + m_n v_n - [m_e(v_e{}' + v_e^\star) + m_i v_i]\Big), \end{aligned} \tag{9}$$

where δ denotes the Dirac measure, and Δ the ionization energy (which is a constant). Notice that the factor 2 in Eq. (7a) is a consequence of the indistinguishability of the two electrons in the right hand side of Eq. (5). This indistinguishability and the principle of detailed balance imply that

$$\sigma^r = \sigma^r(v_e{}', v_e^\star; v_e) = \sigma^r(v_e^\star, v_e{}'; v_e) = \sigma^r(v_e, v_e^\star; v_e{}') = \sigma^{r\prime}. \tag{10}$$

The reference values of the problem are now introduced in order to scale the kinetic system (1).

3. THE SCALED KINETIC SYSTEM

Let ε denote the parameter measuring the relative smallness of the electron mass with respect to the neutral particle:

$$\varepsilon = \sqrt{\frac{m_e}{m_n}} = \sqrt{\frac{m_e}{m_i + m_e}} << 1.$$

Assumption 3.1. We assume that electrons, ions and neutral species have temperatures of the same order of magnitude T_0. Their reference velocities $(v_\alpha)_0$ will be defined from the thermal velocity,

$$(v_\alpha)_0 = \sqrt{\frac{kT_0}{m_\alpha}}, \quad \text{with} \quad \alpha = e, i, n,$$

k being the Boltzmann constant. Consequently, these velocities only depend on the masses, and more precisely we have:

$$(v_n)_0 = \sqrt{1-\varepsilon^2}\ (v_i)_0 = \varepsilon\ (v_e)_0.$$

Besides, we will choose $x_0 = t_0\ (v_e)_0$ as reference length. The reference time t_0 is specified latter on.

Assumption 3.2. We also assume that the densities of the charged particles have the same order of magnitude: $(\rho_e)_0 = (\rho_i)_0$. We introduce a second small parameter measuring the ionization level,

$$\delta = \frac{(\rho_e)_0}{(\rho_n)_0} = \frac{(\rho_i)_0}{(\rho_n)_0},$$

where $(\rho_n)_0$ is the typical density of neutral particles. The distribution function scales are determined from the previous characteristic quantities according to $(f^\alpha)_0 = (\rho_\alpha)_0 (v_\alpha)_0^{-3}$ where $\alpha = e, i, n$.

Assumption 3.3. The force field F_α with $\alpha = e, i, n$, is assumed to derive from a potential and to be relatively weak, i.e. the force term $(F_\alpha m_\alpha^{-1}) \cdot \nabla_{v_\alpha} f^\alpha$ involved in (1) is supposed to be of lower order than the collision operator $(\partial_t f^\alpha)_c$. The force scale is related to the length scale by $(F_\alpha)_0 = kT_0 x_0^{-1}$.

To specify the typical values of the elastic collision operators $Q^{\alpha\beta}$, let us first introduce the following assumptions:

Assumption 3.4. The interaction potentials associated with binary elastic collisions between the various species of charged particles have the same order of magnitude:

$$\sigma_{\alpha\beta}^{\mathcal{F}} = \sigma_0^{\mathcal{F}} \quad \text{for any } \alpha, \beta = e, i.$$

A similar property is supposed to apply to elastic binary collisions involving (at least) one neutral particle, thus:

$$\sigma_{n\alpha}^{\mathcal{B}} = \sigma_{\alpha n}^{\mathcal{B}} = \sigma_0^{\mathcal{B}} \quad \text{where } \alpha = e, i \text{ or } n.$$

Due to Assumption 2.5 the collision cross sections $\sigma_{\alpha\beta}^{\mathcal{F}}$ (resp. $\sigma_{\alpha\beta}^{\mathcal{B}}$) associated with the different species will only differ by the masses through the reduced kinetic energy, as in ref. 10. We can thus distinguish collisions between particles of mass characterized by similar or distinct orders of magnitude to define the scaled scattering cross sections:

$$\sigma_{\alpha\beta}^{\mathcal{B}} = \sigma_{\alpha\beta}^{\mathcal{B}}(\mathcal{E},\chi) = \begin{cases} \sigma_0^{\mathcal{B}}\ \bar{\sigma}^{\mathcal{B}}\Big(\dfrac{\mathcal{E}}{kT_0},\chi\Big), & \text{if } \alpha\neq\beta, \\ \sigma_0^{\mathcal{B}}\ \bar{\sigma}_\star^{\mathcal{B}}\Big(\dfrac{2\mathcal{E}}{kT_0},\chi\Big), & \text{if } \alpha=\beta, \end{cases} \tag{11}$$

and

$$\sigma_{\alpha\beta}^{\mathcal{F}} = \sigma_{\alpha\beta}^{\mathcal{F}}(\mathcal{E}) = \begin{cases} \sigma_0^{\mathcal{F}}\ \bar{\sigma}^{\mathcal{F}}\Big(\dfrac{\mathcal{E}}{kT_0}\Big), & \text{if } \alpha\neq\beta, \\ \sigma_0^{\mathcal{F}}\ \bar{\sigma}_\star^{\mathcal{F}}\Big(\dfrac{2\mathcal{E}}{kT_0}\Big), & \text{if } \alpha=\beta. \end{cases} \tag{12}$$

Denoting by $\tau_{\alpha\beta}$ [resp. $\big(Q^{\alpha\beta}\big)_0$] the characteristic collision time [resp. characteristic elastic collision operator] of a particle of β species against a particle of α species, we have: $\big(Q^{\alpha\beta}\big)_0 = (f_\alpha)_0/\tau_{\alpha\beta}$. These characteristic collision times are given in the Fokker–Planck case (i.e. for $\alpha,\beta\in\{e,i\}$) by

$$\tau_{\alpha\beta} = \left(\frac{m_\alpha}{\mu_{\alpha\beta}}\right)^2 \frac{1}{\sigma_0^{\mathcal{F}}(\rho_\beta)_0} \frac{(v_\alpha)_0}{\Big[\mathrm{Max}\big((v_\alpha)_0,(v_\beta)_0\big)\Big]^3} \mathrm{Min}\left((v_\alpha)_0, \frac{m_\beta}{m_\alpha}(v_\beta)_0\right),$$

and in the Boltzmann case (*i.e.* if $n\in\{\alpha,\beta\}$) by

$$\tau_{\alpha\beta} = \frac{1}{\sigma_0^{\mathcal{B}}(\rho_\beta)_0}\frac{1}{\mathrm{Max}\big((v_\alpha)_0,(v_\beta)_0\big)}.$$

In particular, we have:

$$\tau_{ee} = \frac{1}{\sigma_0^{\mathcal{F}}\ (\rho_e)_0\ (v_e)_0}, \quad \tau_{en} = \frac{1}{\sigma_0^{\mathcal{B}}\ (\rho_n)_0\ (v_e)_0}.$$

This also gives the following orderings:

$$\tau_{ee} = (1-\varepsilon^2)^2 \tau_{ei} = \varepsilon(1-\varepsilon^2)^{3/2} \tau_{ie} = \frac{\varepsilon}{\sqrt{1-\varepsilon^2}} \tau_{ii},$$

and

$$\tau_{en} = \frac{\varepsilon}{\sqrt{1-\varepsilon^2}} \tau_{in} = \varepsilon \tau_{nn} = \delta \tau_{ne} = \frac{\varepsilon}{\sqrt{1-\varepsilon^2}} \delta \tau_{ni}.$$

To conclude the determination of the reference values, the typical value of the ionization-recombination operator $Q^{\alpha,ir}$ can be written $\left(Q^{\alpha,ir}\right)_0 = (\rho_e)_0\ (v_\alpha)_0^{-3}\ \tau_{ir}^{-1}$ where τ_{ir} is the relaxation time of ionizing collisions between electron and neutral particle.

The variables involved in (1)–(2) can now be defined from the previous quantities according to

$$t = t_0\ \bar{t}, \quad x = x_0\ \bar{x}, \quad v_\alpha = (v_\alpha)_0\ \bar{v}_\alpha, \quad f^\alpha = (f^\alpha)_0\ \bar{f}^\alpha, \ldots.$$

In the sequel we will only use reference values and dimensionless variables. To simplify the notations, the bar above dimensionless variables will therefore be omitted from now on.

The dimensionless version of the system of Boltzmann equations (1)–(2) can then be written as

$$\begin{aligned}
&\partial_t f^e + v_e \cdot \nabla_x f^e + F_e \cdot \nabla_{v_e} f^e \\
&\quad = \frac{t_0}{\tau_{ir}} Q^{e,ir}(f^e, f^i, f^n) + \frac{t_0}{\tau_{ee}} \Big[Q^{ee}(f^e, f^e) + Q^{ei}_\varepsilon(f^e, f^i) \Big] \\
&\qquad + \frac{t_0}{\tau_{en}} Q^{en}_\varepsilon(f^e, f^n), \\
&\partial_t f^i + \frac{\varepsilon}{\sqrt{1-\varepsilon^2}} \Big(v_i \cdot \nabla_x f^i + F_i \cdot \nabla_{v_i} f^i \Big) \\
&\quad = \frac{t_0}{\tau_{ir}}\ Q^{i,ir}(f^e, f^i, f^n) + \frac{t_0}{\tau_{ee}} \Big[\frac{\varepsilon}{\sqrt{1-\varepsilon^2}}\ Q^{ii}(f^i, f^i) + \varepsilon\ Q^{ie}_\varepsilon(f^i, f^e) \Big] \\
&\qquad + \frac{t_0}{\tau_{en}}\ \frac{\varepsilon}{\sqrt{1-\varepsilon^2}}\ Q^{in}_\varepsilon(f^i, f^n), \\
&\partial_t f^n + \varepsilon \Big(v_n \cdot \nabla_x f^n + F_n \cdot \nabla_{v_n} f^n \Big) \\
&\quad = \frac{t_0}{\tau_{ir}}\ \delta\ Q^{n,ir}(f^e, f^i, f^n) + \frac{t_0}{\tau_{en}} \Big[\varepsilon\ Q^{nn}(f^n, f^n) + \varepsilon\ \delta\ Q^{ne}_\varepsilon(f^n, f^e) \\
&\qquad + \frac{\varepsilon\ \delta}{\sqrt{1-\varepsilon^2}}\ Q^{ni}_\varepsilon(f^n, f^i) \Big].
\end{aligned} \tag{13}$$

The scaled collision operators are now detailed. In the Boltzmann case, we have (note that the factor $1/\varepsilon$ just below is due to the fact that the integral term in the expression of Q_ε^{ne} is of order ε; we refer to[10] for details, and to Lemma 4.1 below):

$$Q_\varepsilon^{ne}(f^n, f^e)(v_n) = \frac{\sqrt{1+\varepsilon^2}}{\varepsilon}\int_{IR^3\times S^2} B^{\mathcal{B}}\Big(\frac{\varepsilon v_n - v_e}{\sqrt{1+\varepsilon^2}}, \Omega\Big)\Big(f_\varepsilon^{n\prime} f_\varepsilon^{e\prime} - f^n f^e\Big) dv_e\, d\Omega,$$

$$Q_\varepsilon^{en}(f^e, f^n)(v_e) = \sqrt{1+\varepsilon^2}\int_{IR^3\times S^2} B^{\mathcal{B}}\Big(\frac{v_e - \varepsilon v_n}{\sqrt{1+\varepsilon^2}}, \Omega\Big)\Big(f_\varepsilon^{e\prime} f_\varepsilon^{n\prime} - f^e f^n\Big) dv_n\, d\Omega,$$

$$Q_\varepsilon^{ni}(f^n, f^i)(v_n) = \sqrt{1-\tfrac{1}{2}\varepsilon^2}\int_{IR^3\times S^2} B_\star^{\mathcal{B}}\Big(\frac{\sqrt{1-\varepsilon^2}v_n - v_i}{\sqrt{1-\frac{1}{2}\varepsilon^2}}, \Omega\Big)\Big(f_\varepsilon^{n\prime} f_\varepsilon^{i\prime} - f^n f^i\Big) dv_i d\Omega,$$

$$Q_\varepsilon^{in}(f^i, f^n)(v_i) = \sqrt{1-\tfrac{1}{2}\varepsilon^2}\int_{IR^3\times S^2} B_\star^{\mathcal{B}}\Big(\frac{v_i - \sqrt{1-\varepsilon^2}v_n}{\sqrt{1-\frac{1}{2}\varepsilon^2}}, \Omega\Big)\Big(f_\varepsilon^{i\prime} f_\varepsilon^{n\prime} - f^i f^n\Big) dv_n d\Omega,$$

and

$$Q^{nn}(f^n, f^n)(v_n) = \int_{IR^3\times S^2} B_\star^{\mathcal{B}}\left(v_n - v_n^\star, \Omega\right)\Big(f^{n\prime} f^{n\star\prime} - f^n f^{n\star}\Big) dv_n^\star\, d\Omega.$$

The dimensionless kernels $B^{\mathcal{B}}(v,\Omega)$ and $B_\star^{\mathcal{B}}(v,\Omega)$ are set equal to zero when Ω satisfies $(v\cdot\Omega) < 0$ (i.e. when χ lies within the range [−1,0]). When Ω belongs to S_+^2, and thus $\chi \in [0,1]$, they are expressed in term of the related dimensionless scattering cross sections, as follows:

$$\begin{aligned} B^{\mathcal{B}}(v,\Omega) &= B^{\mathcal{B}}(|v|,\chi) = |v|\ \bar\sigma^{\mathcal{B}}(|v|^2,\chi), \qquad \text{with } |v|^2 = \tfrac{\mathcal{E}}{kT_0} \\ B_\star^{\mathcal{B}}(v,\Omega) &= B_\star^{\mathcal{B}}(|v|,\chi) = |v|\ \bar\sigma_\star^{\mathcal{B}}(|v|^2,\chi), \qquad \text{with } |v|^2 = \tfrac{2\mathcal{E}}{kT_0}. \end{aligned} \tag{14}$$

Concerning the Fokker–Planck–Landau case, the scaled collision operators read:

$$\begin{aligned} Q^{\alpha\alpha}(f^\alpha, f^\alpha)(v_\alpha) &= \nabla_{v_\alpha}\cdot\int_{IR^3} B_\star^{\mathcal{F}}\left(v_\alpha - v_\alpha^\star\right)\ S\left(v_\alpha - v_\alpha^\star\right) \\ &\quad \times\left(\nabla_{v_\alpha} f^\alpha f_\star^\alpha - \nabla_{v_\alpha^\star} f_\star^\alpha f^\alpha\right)\ dv_\alpha^\star \quad \text{with } \alpha = e, i, \\ Q_\varepsilon^{ei}(f^e, f^i)(v_e) &= \sqrt{1-\varepsilon^2} \\ &\quad \times\nabla_{v_e}\cdot\int_{IR^3} B^{\mathcal{F}}\left(\sqrt{1-\varepsilon^2}\ v_e - \varepsilon\ v_i\right) S\left(v_e - \frac{\varepsilon}{\sqrt{1-\varepsilon^2}}\ v_i\right) \\ &\quad \times\left(\nabla_{v_e} f^e f^i - \frac{\varepsilon}{\sqrt{1-\varepsilon^2}}\ \nabla_{v_i} f^i f^e\right)\ dv_i, \end{aligned}$$

and

$$Q_\varepsilon^{ie}(f^i, f^e)(v_i) = -\nabla_{v_i} \cdot \int_{IR^3} B^{\mathcal{F}} \left(\varepsilon v_i - \sqrt{1-\varepsilon^2} v_e\right) \\ \times S\left(\frac{\varepsilon}{\sqrt{1-\varepsilon^2}} v_i - v_e\right) \times \left(\nabla_{v_e} f^e f^i - \frac{\varepsilon}{\sqrt{1-\varepsilon^2}} \nabla_{v_i} f^i f^e\right) dv_e.$$

The dimensionless kernels $B^{\mathcal{F}}(v)$ and $B_\star^{\mathcal{F}}(v)$ are also defined from the related scaled scattering cross sections of (12), according to

$$\begin{aligned} B^{\mathcal{F}}(v) &= B^{\mathcal{F}}(|v|) = |v|^3 \ \bar{\sigma}^{\mathcal{F}}(|v|^2), \\ B_\star^{\mathcal{F}}(v) &= B_\star^{\mathcal{F}}(|v|) = |v|^3 \ \bar{\sigma}_\star^{\mathcal{F}}(|v|^2). \end{aligned} \tag{15}$$

Notice that with these definitions we have $Q_\varepsilon^{ne} = \mathcal{O}(1)$ as well as $Q_\varepsilon^{ie} = \mathcal{O}(1)$.

Remark. In the particular case of elastic collisions involving electrons and neutral particles, the previous scaled operators are exactly the same as in ref. 10. They differ from that latter work concerning collisions with ions, due to the definition of ε. But the derivations remain similar. The reader can thus refer to ref. 10 for further details.

The scaled versions of the ionization-recombination operators are given by Eqs. (7a)–(7c) where $\mathcal{F}_0$ now denotes the dimensionless quantity $\mathcal{F}_0(\bar{f}^e)^{-1}$. The scaled conservation Eqs. (9) are written:

$$\begin{aligned} \delta_{\mathcal{E}} &= \delta\left(|v_e|^2 + |v_n|^2 - [|v_e'|^2 + |v_e^\star|^2 + |v_i|^2 + 2\Delta]\right), \\ \delta_v &= \delta\left(\varepsilon \ v_e + v_n - [\varepsilon \ (v_e' + v_e^\star) + \sqrt{1-\varepsilon^2} \ v_i]\right), \end{aligned} \tag{16}$$

where Δ holds for the ionization energy scaled by the thermal energy kT_0.

Remark. Let us underline that choosing the time scaling

$$t_0 = \tau_{en} = \varepsilon^2 \tau_{ir} = \delta \ \tau_{ee} \quad \text{while} \quad \delta \to 0,$$

we recover the dimensionless system studied in ref. 11. This corresponds to a very weakly ionized plasma, such as a glow discharge, where ionization occurs very seldom and the ionization level δ lies within the range 10^{-8} to 10^{-5}, such that $\delta \ll \varepsilon$.

Assumption 3.5. In the present study, we assume that

$$t_0 = \tau_{en} = \varepsilon\ \tau_{ir} = \delta\ \tau_{ee} \quad \text{and} \quad \delta = \varepsilon. \tag{17}$$

We will indeed investigate a plasma partially ionized, such as an arc discharge. The ionization level δ is then several orders of magnitude larger than in ref. 11 since it lies within the range 10^{-3} to 10^{-1}, implying that $\delta \approx \varepsilon$. Besides, within this framework, impact ionization gets a leading order collisional process.

Thus the scaled system of kinetic equations writes:

$$\begin{aligned}
\partial_t f^e + v_e \cdot \nabla_x f^e + F_e \cdot \nabla_{v_e} f^e &= Q_\varepsilon^{en}(f^e, f^n) \\
&\quad +\varepsilon \big[Q^{ee}(f^e, f^e) \\
&\quad + Q_\varepsilon^{ei}(f^e, f^i) + Q^{e,ir}(f^e, f^i, f^n) \big], \\
\partial_t f^i + \frac{\varepsilon}{\sqrt{1-\varepsilon^2}} \left(v_i \cdot \nabla_x f^i + F_i \cdot \nabla_{v_i} f^i \right) &= \frac{\varepsilon}{\sqrt{1-\varepsilon^2}} Q_\varepsilon^{in}(f^i, f^n) \\
&\quad +\varepsilon Q^{i,ir}(f^e, f^i, f^n) \\
&\quad +\varepsilon^2 \left[\frac{1}{\sqrt{1-\varepsilon^2}} Q^{ii}(f^i, f^i) + Q_\varepsilon^{ie}(f^i, f^e) \right], \\
\partial_t f^n + \varepsilon \left(v_n \cdot \nabla_x f^n + F_n \cdot \nabla_{v_n} f^n \right) &= \varepsilon Q^{nn}(f^n, f^n) \\
&\quad +\varepsilon^2 \left[Q_\varepsilon^{ne}(f^n, f^e) + \frac{1}{\sqrt{1-\varepsilon^2}} Q_\varepsilon^{ni}(f^n, f^i) \right. \\
&\quad \left. + Q^{n,ir}(f^e, f^i, f^n) \right].
\end{aligned} \tag{18}$$

Remark. Collisions of electrons, ions and neutral particles with neutral particles are leading order collisional processes, as in any weakly ionized plasma. However, contrary to the very weakly ionized problem studied in ref. 11, the leading order operators of neutral particles and ions are also coupled with electrons. Besides, the collision operators for elastic electron collisions Q^{ee} and ionization $Q^{e,ir}$ have now the same order of magnitude. But the main novelty comes from the ionization-recombination operator for ions, which is here among the leading order operators. As we will see later, the ionization-recombination operator is also crucial for the determination of the equilibrium state for electrons and for the fluid limit: in fact, if it were of less order (such as in ref. 11), the equilibrium state for the electrons would be only isotropic (and not Maxwellian), and the fluid model for the electrons would be a Spherical Harmonics Expansion-model (or SHE-model, refs. 15,16 for instance). In this model, the equilibrium function is an isotropic function of the velocity v, and it satisfies a diffusion equation (with respect to the time and space variables) which is parametrized by the energy $W = |v|^2/2$. If in addition, we suppose that this distribution function is Maxwellian, then we can recover a classical hydrodynamic model (of diffusive type) by taking the moments of the previous equation with respect to W (see also ref. 17 for plasmas).

Before going on, let us now briefly explain the way we proceed in this paper, in order to derive a fluid model (stated in Theorem 5.6 below) for the mixture.

The starting point of the analysis is the coupled system of scaled transport-collision equations (18). The method (detailed in paragraph 5), is based on a classical Hilbert expansion, which consists in first doing a formal asymptotic expansion of each distribution function f^e, f^i, f^n in terms of the small parameter ε (according to (35)), and then identifying terms of equal powers in the scaled kinetic equations (18). This supposes we have first expanded the collision operators in the right hand side of these equations in terms of ε; concerning the interspecies elastic collision operators, this has been investigated in refs. 10,18 within the frame of a mixture made of two species of disparate masses. In Lemmas 4.1 and 4.2 below, we recall and extend these results to our case, for which we have three species: electrons, ions and neutrals.

The zero order terms in the expansion of the distribution functions, respectively denoted by f_0^e, f_0^i and f_0^n, are given in Theorem 5.1: they are all classical Maxwellian distributions, and f_0^i and f_0^n have same mean velocity (denoted by u) and temperature (T); they thus only differ through their densities, which are respectively denoted by ρ^i and ρ^n. Moreover, f_0^e has zero mean velocity, and its density ρ_e and temperature T_e are linked to the density of the two other species (ρ^i and ρ^n) by a generalized Saha law (25). As we will see below, the way we derive these equilibrium is not classical for the charged particles, due to the presence of strong ionization-recombination processes.

Some technical lemmas are needed, in order to derive these results. We first show that f_0^e satisfies a linear homogeneous equation, which kernel is made of isotropic functions (see Lemma 4.3). Then we find that f_0^i artificially solves a linear equation (studied in Lemma 4.4), which right hand side (which is directly linked to the ionization processes) is in fact zero. This means that, up to this order, there is no production (or annihilation) of ions due to ionization; and there is also naturally no production (or annihilation) of electrons. This last fact, combined to some weak properties of the collision term $Q^{e,ir}$ (entropy inequality) stated in Lemma 4.5, allow to derive the final expression for f_0^e as a centered Maxwellian, which density and temperature are linked to the densities of the two other species.

In order to derive a fluid model for the macroscopic quantities we have exhibited, we have to go on in the identification, and compute corrections of order one for each distribution function (denoted by f_1^e, f_1^i and f_1^n). For this, we need to define the expansion of the ionization-recombination operators in terms of ε: this is achieved around equilibrium states in Lemma 4.6.

The next paragraph is devoted to all these technical lemmas we have mentioned, which are devoted to the different collision operators: asymptotic expansion in terms of ε, solvability conditions for the linear operators involved in the different steps of the identification, entropy inequalities, etc. The Hilbert method itself and the fluid limit are detailed in paragraph 5.

4. MAIN PROPERTIES OF THE COLLISION OPERATORS

Let us first give expansions, with respect to the small parameter ε, of the elastic collision operators $Q_\varepsilon^{\alpha\beta}$. We also recall some elementary properties of the different terms of these expansions: parity, mass conservation. In the Boltzmann case, we have:

Lemma 4.1. Let f^α, where $\alpha = e, i, n$, be sufficiently regular functions.

(i) Let α, $\beta = i$, n and $\alpha \neq \beta$. Then

$$Q_\varepsilon^{\alpha\beta}(f^\alpha, f^\beta) = Q_0^{\alpha\beta}(f^\alpha, f^\beta) + \mathcal{O}(\varepsilon^2),$$

with

$$Q_0^{\alpha\beta}(f^\alpha, f^\beta)(v_\alpha) = \int_{I\!R^3\times S^2} B_\star^{\mathcal{B}}\left(v_\alpha - v_\beta^\star, \Omega\right)\left(f^{\alpha'} f_\star^{\beta'} - f^\alpha f_\star^\beta\right) dv_\beta^\star \, d\Omega.$$

(ii) Let α, $\beta = e$, n and $\alpha \neq \beta$. Then

$$Q_\varepsilon^{\alpha\beta}(f^\alpha, f^\beta) = Q_0^{\alpha\beta}(f^\alpha, f^\beta) + \varepsilon \; Q_1^{\alpha\beta}(f^\alpha, f^\beta) + \mathcal{O}(\varepsilon^2),$$

with

$$\begin{aligned} Q_0^{en}(f^e, f^n)(v_e) & \\ &= q_e^{\mathcal{B}}(f^e)(v_e) \int_{I\!R^3} f^n(v_n) dv_n, \\ Q_1^{en}(f^e, f^n)(v_e) &= \left(-\nabla_{v_e}[q_e^{\mathcal{B}}(f^e)] + q_e^{\mathcal{B}}(\nabla_{v_e} f^e)\right)(v_e) \cdot \int_{I\!R^3} v_n f^n(v_n) dv_n, \\ Q_0^{ne}(f^n, f^e)(v_n) &= -2\nabla_{v_n} f^n(v_n) \cdot \int_{I\!R^3\times S^2} B^{\mathcal{B}}(v_e, \Omega) \frac{(v_e\cdot\Omega)^2}{|v_e|^2} v_e f^e(v_e) dv_e d\Omega, \end{aligned}$$

$$
\begin{aligned}
Q_1^{ne}(f^n,f^e)(v_n) = 2\nabla_{v_n}^2 f^n(v_n)
& : \Bigg[\int_{I\!R^3\times S^2} B^{\mathcal{B}}(v_e,\Omega)\frac{(v_e,\Omega)^4}{|v_e|^4}(v_e\otimes v_e) f^e(v_e)dv_e d\Omega \\
& + \frac{1}{2}\int_{I\!R^3\times S^2} B^{\mathcal{B}}(v_e,\Omega)(v_e,\Omega)^2\left(1-\frac{(v_e,\Omega)^2}{|v_e|^2}\right) \\
& \times S(v_e)\ f^e(v_e)\ dv_e\ d\Omega\Bigg] - 2[\nabla_{v_n}(v_n f^n)]^s(v_n) \\
& : \int_{I\!R^3\times S^2} B^{\mathcal{B}}(v_e,\Omega)\frac{(v_e,\Omega)^2}{|v_e|^2}(v_e\otimes\nabla_{v_e} f^e)^s(v_e)\ dv_e\ d\Omega.
\end{aligned}
$$

The superscript s indicates that a tensor is symmetrized, and the notation $A:B$, where A and B are two matrices with respective entries A_{ij}, B_{ij}, denotes the contracted product: $\sum_{i,j} A_{ij}B_{ij}$. Moreover, the linear operator $q_e^{\mathcal{B}}$ is defined by

$$
q_e^{\mathcal{B}}(f^e)(v_e) = \int_{S^2} B^{\mathcal{B}}(v_e,\Omega)\left[f^e\left(v_e - 2(v_e,\Omega)\Omega\right) - f^e(v_e)\right]d\Omega.
$$

(iii) For any f^e, f^n we have:

$$
Q_i^{en}\left[f^e(-v_e), f^n\right](v_e) = (-1)^i Q_i^{en}\left[f^e, f^n\right](-v_e)
$$

(iv) Mass conservation implies that:

$$
\int_{I\!R^3} Q_j^{en}\Big(f^e, f^n\Big)(v_e)\ dv_e = 0, \qquad \forall j \in I\!N.
$$

Proof. The expansions of Q_ε^{en} and Q_ε^{ne} are identical to the expressions proposed in ref. 10. Besides, the other expansions specific to the present problem are obtained in a similar way. Points (iii) and (iv) is a particular case of a result stated in ref. 10, propositions 4.8 and 4.7. ∎

In the Fokker–Planck–Landau case, we obtain:

Lemma 4.2.

(i) Let f^α with $\alpha = e, i$, be sufficiently regular functions. Then

$$
\begin{aligned}
Q_\varepsilon^{ei}(f^e,f^i) &= Q_0^{ei}(f^e,f^i) + \varepsilon\ Q_1^{ei}(f^e,f^i) + \mathcal{O}(\varepsilon^2), \\
Q_\varepsilon^{ie}(f^i,f^e) &= Q_0^{ie}(f^i,f^e) + \mathcal{O}(\varepsilon),
\end{aligned}
$$

with

$$Q_0^{ei}(f^e, f^i)(v_e) = q_e^{\mathcal{F}}(f^e)(v_e) \int_{I\!R^3} f^i(v_i) dv_i,$$
$$Q_1^{ei}(f^e, f^i)(v_e) = \left(-\nabla_{v_e}[q_e^{\mathcal{F}}(f^e)] + q_e^{\mathcal{F}}(\nabla_{v_e} f^e)\right)(v_e) \cdot \int_{I\!R^3} v_i f^i(v_i) dv_i,$$
$$Q_0^{ie}(f^i, f^e)(v_i) = -2\ \nabla_{v_i} f^i(v_i) \cdot \int_{I\!R^3} \frac{B^{\mathcal{F}}(v_e)}{|v_e|^2} v_e\ f^e(v_e)\ dv_e,$$

and

$$q_e^{\mathcal{F}}(f^e) = \nabla_{v_e} \cdot \left[B^{\mathcal{F}}\, S\, \nabla_{v_e} f^e \right].$$

(ii) Mass conservation implies that:

$$\int_{I\!R^3} Q_j^{ei}\left(f^e, f^i\right)(v_e)\, dv_e = 0, \quad \forall j \in I\!N.$$

Proof. Again, details of the derivations are not reported here since a similar result is stated in ref. 10. ∎

We now examine some properties of the linear operators involved in the different steps of the Hilbert expansion. In the sequel, we denote by $M_{\mathrm{u}_\alpha, T_\alpha}$ the normalized (i.e. with mean density equal to 1) Maxwellian of mean velocity u_α and temperature T_α defined by

$$M_{\mathrm{u}_\alpha, T_\alpha}(v) = \frac{1}{(2\pi T_\alpha)^{3/2}} \exp\left[-\frac{(v - \mathrm{u}_\alpha)^2}{2T_\alpha} \right]. \tag{19}$$

First, concerning the electrons, the linear operator involved is the operator L_{en} defined by

$$L_{en}\phi = Q_0^{en}(\phi, f_0^n) = \rho_n\ q_e^{\mathcal{B}}(\phi), \tag{20}$$

where $\rho_n = \int_{I\!R^3} f_0^n(v_n) dv_n$ is the density of neutral particles.
Let us first recall a result of ref. 19.

Lemma 4.3. (i) The kernel of the operator L_{en} is made of isotropic functions, i.e. functions $\phi = \phi(v)$ such that $\phi(v) = \bar{\phi}(|v|)$. In particular, if φ is an odd function of the velocity variable, then the equation $L_{en}\phi = \varphi$ has a unique odd solution ϕ_0 and any other solution ϕ writes $\phi = \phi_0 + \bar{\phi}$, where $\bar{\phi}$ is isotropic.

(ii) More generally, let us introduce the energy variable $W(v)=|v|^2/2$, and the sphere $S_W=\{v\in I\!R^3,\ W(v)=W\}$; we recall the co-area formula

$$\int_{I\!R^3} f(v)dv=\int_0^{+\infty}\left(\int_{S_W} f(v)\,dN(v)\right)dW,$$

where $dN(v)=\frac{dS_W(v)}{|\nabla W(v)|}=\frac{dS_W(v)}{\sqrt{2W}}$ (dS_W is the euclidian surface element on S_W). Then the equation $L_{en}\phi=\varphi$ has a solution if and only if the right hand side satisfies the following orthogonality relation:

$$\forall W>0,\quad \int_{S_W}\varphi(v)\,dN(v)=0. \tag{21}$$

In particular, for any $T>0$, the relation:

$$\int_{I\!R^3}\varphi(v)\,M_{0,T}(v)\,dv=0, \tag{22}$$

is a necessary condition of solvability.

Remark. The same result holds in the Fokker–Planck case, i.e. for the operator $q_e^{\mathcal{F}}$ defined in Lemma 4.2 (see ref. 17). This property is characteristic of what is usually called Lorentz operators.
For the ions, we define the linear operator L_{in} by

$$L_{in}\phi=M_{u,T}^{-1}\,Q_0^{in}\left(M_{u,T}\phi,\,\rho_n M_{u,T}\right), \tag{23}$$

with Q_0^{in} given in Lemma 4.1. We have

$$L_{in}\phi(v_i)=\rho_n\int_{I\!R^3\times S^2} B_\star^{\mathcal{B}}(v_i-v_n,\Omega)\ M_{u,T}(v_n)\ \left[\phi(v_i{}')-\phi(v_i)\right]dv_n\ d\Omega,$$

with the notation $v_i{}'=v_i-(v_i-v_n,\Omega)\Omega$. Then

Lemma 4.4. The linear operator L_{in} is self-adjoint on the weighted Hilbert space defined by

$$L^2_{M_{u,T}}=\left\{f\ /\ \int_{I\!R^3} f^2(v)\ M_{u,T}(v)\ dv\ <\ +\infty\right\},$$

and its kernel is made of constant functions. Moreover, under suitable assumptions on $B_\star^{\mathcal{B}}$, the equation $L_{in}\phi=\varphi$ is solvable if and only if the right hand side φ satisfies the orthogonality relation:

$$\int_{I\!R^3} \varphi(v)\ M_{u,T}(v)\ dv=0;$$

the solution ϕ is then unique, up to an additive constant.

Proof. The proof partly results from the following weak formulation

$$\begin{aligned}
&\int_{I\!R^3} L_{in}\phi(v_i)\varphi(v_i)\ M_{u,T}(v_i)\ dv_i \\
&\quad=-\frac{1}{2}\rho_n \int_{I\!R^3}\int_{I\!R^3\times S^2} B_\star^{\mathcal{B}}(v_i-v_n,\Omega)\left[\phi(v_i{}')-\phi(v_i)\right] \\
&\qquad\times\left[\varphi(v_i{}')-\varphi(v_i)\right]\ M_{u,T}(v_i)\ M_{u,T}(v_n)\ d\Omega\ dv_n\ dv_i. \qquad \blacksquare
\end{aligned}$$

We now turn investigating some properties of the ionization-recombination collision operator: in Lemma 4.5 below, we first state a weak formulation, and an entropy inequality, for the dominating part of this operator (in terms of ε), while Lemma 4.5 is devoted to the computation of its linearization. For any $\alpha\in\{e,i,n\}$, we simply denote by $Q_0^{\alpha,ir}$ the limit, when ε goes to zero, of the operator $Q^{\alpha,ir}$. These three leading order ionization-recombination operators are still given by expression (7a)–(7c), but where the conservation Eqs. (16) have to be replaced by their limit when ε goes to zero, *i.e.* by

$$\begin{aligned}
\delta_v &= \delta(v_n-v_i), \\
\delta_{\mathcal{E}} &= \delta\left(|v_e|^2-[|v_e{}'|^2+|v_e^\star|^2+2\Delta]\right),
\end{aligned} \tag{24}$$

In the sequel, the most important (and useful) result concerns the electrons. We have:

Lemma 4.5. Let $f^n=\rho_n M_{u,T}$, $f^i=\rho_i M_{u,T}$ and $\delta_{\mathcal{E}}$ be defined by (24).

(i) Then, for any regular test function ψ, we have the following weak formulation:

$$\begin{aligned}
&\int_{I\!R^3} Q_0^{e,ir}\left(f^e,f^i,f^n\right)(v_e)\ \psi(v_e)\ dv_e= \\
&-\rho_i\int_{I\!R^9}\sigma^r\ \delta_{\mathcal{E}}\left[\mathcal{F}_0\ \frac{\rho_n}{\rho_i}\ f^e(v_e)-f^e(v_e{}')\ f^e(v_e^\star)\right] \\
&\times\left[\psi(v_e)-\psi(v_e{}')-\psi(v_e^\star)\right]\ dv_e\ dv_e{}'\ dv_e^\star.
\end{aligned}$$

(ii) Let $H(f^e) = \log(\mathcal{F}_0^{-1}\ \rho_n^{-1}\ \rho_i\ f^e)$, and let σ^r be positive, then

$$\begin{aligned}
&\int_{I\!R^3} Q_0^{e,ir}\Big(f^e, f^i, f^n\Big)(v_e)\ H(f^e)(v_e)\ dv_e \\
&\quad = -\rho_i \int_{I\!R^9} \sigma^r\ \delta_{\mathcal{E}} \left[\mathcal{F}_0\ \frac{\rho_n}{\rho_i}\ f^e(v_e) - f^e(v_e{}')\ f^e(v_e^\star)\right] \\
&\qquad \times \left[\log\left(\mathcal{F}_0 \frac{\rho_n}{\rho_i}\ f^e(v_e)\right) - \log\left(f^e(v_e{}') f^e(v_e^\star)\right)\right] dv_e dv_e{}' dv_e^\star \leqslant 0.
\end{aligned}$$

(iii) In particular, if f^e is isotropic (*i.e.* $f^e(v^e) = f^e(|v^e|)$) *and such that* $Q_0^{e,ir}\Big(f^e, f^i, f^n\Big) = 0$, *then* $f^e = \rho_e M_{0,T_e}$, *with:*

$$\rho_e = \frac{\mathcal{F}_0\ \rho_n}{\rho_i}\ (2\pi T_e)^{3/2}\ \exp\left(-\frac{\Delta}{T_e}\right). \tag{25}$$

Proof. The proof of the first two points is straightforward using the symmetry property (10) and Eq. (16). We now show point (iii). If f^e is isotropic, the function H defined by $H = \log[(\rho_i f^e)/(\rho_n \mathcal{F}_0)]$, is also isotropic. With the notation $\mathcal{E} = \frac{1}{2}|v_e|^2$, we can introduce the function φ of the energy variable $\mathcal{E}$ by setting: $\varphi(\mathcal{E}) = H(|v_e|)$.
Let us now suppose that: $Q_0^{e,ir}\Big(f^e, f^i, f^n\Big) = 0$; we get in particular:

$$\int_{I\!R^3} Q_0^{e,ir}\Big(f^e, f^i, f^n\Big)(v_e)\ H(f^e)(v_e)\ dv_e = 0.$$

Using the positivity of the recombination cross section σ^r, we deduce from the entropy inequality (ii) that:

$$\varphi(\mathcal{E}) = \varphi(\mathcal{E}') + \varphi(\mathcal{E}^\star) \quad \text{with} \quad \mathcal{E} = \mathcal{E}' + \mathcal{E}^\star + \Delta, \tag{26}$$

whenever $\mathcal{E}, \mathcal{E}', \mathcal{E}^\star \in I\!R^+$. From (26), we have $\mathcal{E} \in [\Delta, \bar{\mathcal{E}})$ and $\mathcal{E}', \mathcal{E}^\star \in [0, \bar{\mathcal{E}} - \Delta)$ where $\bar{\mathcal{E}}$ can be arbitrarily large.
Next, we successively proceed to the following changes of variables in (26): 1, 2) $\mathcal{E} \to \mathcal{E} \pm h$ and $\mathcal{E}^\star$ fixed, (3) $\mathcal{E} \to \mathcal{E} - h$ and $\mathcal{E}'$ fixed, (4) $\mathcal{E}' \to \mathcal{E}' + h$ and $\mathcal{E}$ fixed, and substract (26) to obtain:

$$\begin{aligned}
\varphi(\mathcal{E}' \pm h) - \varphi(\mathcal{E}') &= \varphi(\mathcal{E} \pm h) - \varphi(\mathcal{E}), \\
\varphi(\mathcal{E}^\star) - \varphi(\mathcal{E}^\star - h) &= \varphi(\mathcal{E}) - \varphi(\mathcal{E} - h), \\
\varphi(\mathcal{E}' + h) - \varphi(\mathcal{E}') &= \varphi(\mathcal{E}^\star) - \varphi(\mathcal{E}^\star - h).
\end{aligned}$$

Combining the resulting equations gives

$$\begin{aligned}\varphi(\mathcal{E}+h)-2\ \varphi(\mathcal{E})+\varphi(\mathcal{E}-h) &= 0,\\ \varphi(\mathcal{E}'+h)-2\ \varphi(\mathcal{E}')+\varphi(\mathcal{E}'-h) &= 0.\end{aligned} \tag{27}$$

As h can be made arbitrarily small, Eqs. (27) show that φ is an affine function in the intervals $[\Delta, \bar{\mathcal{E}})$ and $[0, \bar{\mathcal{E}}-\Delta)$, respectively. But $\bar{\mathcal{E}}$ can be arbitrarily large, while Δ is assumed to be finite. There exists thus a value of $\bar{\mathcal{E}}$ such that $\bar{\mathcal{E}}-\Delta \geqslant \Delta$, which implies that the two intervals cover the whole energy interval $I\!R^+$. Consequently, we conclude that $\varphi(\mathcal{E})=a+c\ \mathcal{E}$ with $\mathcal{E}\in I\!R^+$. The determination of a is then straightforward substituting the previous expression of φ in (26). Thus, $H(v^e)=c\ \left(\frac{1}{2}|v_e|^2+\Delta\right)$ and

$$f_0^e(|v_e|)=\frac{\mathcal{F}_0\rho_n}{\rho_i}\ \exp\left[-\frac{|v_e|^2+2\Delta}{2T_e}\right],$$

which concludes the proof. ∎

Let us now precise the expansion, in terms of ε, of each ionization-recombination operator around the equilibrium state.

Lemma 4.6. Let us set: $f_0^\alpha=\rho_\alpha M_{u,T}$ for $\alpha=i,n$ and $f_0^e=\rho_e M_{0,T_e}$. We expand the distributions in terms of ε by setting:

$$f_\varepsilon^\alpha=f_0^\alpha\ \left(1+\varepsilon\ \phi_1^\alpha+\varepsilon^2\phi_2^\alpha\right)+\mathcal{O}(\varepsilon^3)\quad for\quad \alpha=e,i,n. \tag{28}$$

Then, if ρ_e is given by (25), we have, for any $\alpha=e,i,n$:

$$Q_0^{\alpha,ir}\left(f_0^e, f_0^i, f_0^n\right)=0, \tag{29}$$

and

$$\begin{aligned}Q^{\alpha,ir}\left(f_\epsilon^e, f_\epsilon^i, f_\epsilon^n\right) = {}&\epsilon\mathcal{L}Q^{\alpha,ir}\left(\phi_1^e,\phi_1^i,\phi_1^n\right)\\ &+\epsilon^2\left[\mathcal{L}Q^{\alpha,ir}\left(\phi_2^e,\phi_2^i,\phi_2^n\right)+DQ^{\alpha,ir}\left(\phi_1^e,\phi_1^i,\phi_1^n\right)+R^\alpha\right]\\ &+\mathcal{O}(\epsilon^3),\end{aligned} \tag{30}$$

with

$$
\begin{aligned}
&\left(f_0^e(v_e)\right)^{-1} \mathcal{L}Q^{e,ir}\Big(\phi^e,\phi^i,\phi^n\Big)(v_e)\\
&\quad = -\mathcal{F}_0 \int_{I\!R^9} \sigma^r \;\; \delta_{\mathcal{E}} \;\; f_0^n(v_i)\Big[\phi^e(v_e)+\phi^n(v_i)\\
&\qquad\qquad\qquad\qquad -\phi^e({v_e}')-\phi^e(v_e^\star)-\phi^i(v_i)\Big]dv_i dv_e^\star \; d{v_e}'\\
&\qquad +2\int_{I\!R^9} \sigma^r \;\; \delta_{\mathcal{E}'} f_0^e(v_e^\star) \;\; f_0^i(v_i)\Big[\phi^e({v_e}')+\phi^n(v_i)\\
&\qquad\qquad\qquad\qquad -\phi^e(v_e)-\phi^e(v_e^\star)-\phi^i(v_i)\Big]dv_i \;\; dv_e^\star \;\; d{v_e}',\\
&\Big(f_0^i(v_i)\Big)^{-1} \mathcal{L}Q^{i,ir}\Big(\phi^e,\phi^i,\phi^n\Big)(v_i)\\
&\quad = \int_{I\!R^9} \sigma^r \;\; \delta_{\mathcal{E}} f_0^e({v_e}') f_0^e(v_e^\star)\Big[\phi^e(v_e)+\phi^n(v_i)\\
&\qquad\qquad\qquad\qquad -\phi^e({v_e}')-\phi^e(v_e^\star)-\phi^i(v_i)\Big]dv_e \;\; dv_e^\star \;\; d{v_e}',\\
&\left(f_0^n(v_n)\right)^{-1} \mathcal{L}Q^{n,ir}\Big(\phi^e,\phi^i,\phi^n\Big)(v_n)\\
&\quad = \mathcal{F}_0 \int_{I\!R^9} \sigma^r \delta_{\mathcal{E}} f_0^e(v_e)\Big[\phi^e({v_e}')+\phi^e(v_e^\star)+\phi^i(v_n)\\
&\qquad\qquad\qquad\qquad -\phi^e(v_e)-\phi^n(v_n)\Big]dv_e \;\; dv_e^\star \;\; d{v_e}',
\end{aligned}
$$

and

$$
\begin{aligned}
&(f_0^e(v_e))^{-1} DQ^{e,ir}(\phi^e,\phi^i,\phi^n)(v_e)\\
&\quad = -\mathcal{F}_0 \int_{I\!R^9} \sigma^r \;\; \delta_{\mathcal{E}} \;\; f_0^n(v_i)\Big[\phi^e(v_e)\;\;\phi^n(v_i)-\phi^e({v_e}')\;\;\phi^e(v_e^\star)\\
&\qquad\qquad\qquad\qquad -\phi^i(v_i)\;\;\phi^e({v_e}')-\phi^i(v_i)\;\;\phi^e(v_e^\star)\Big]\;\; dv_i dv_e^\star d{v_e}'\\
&\qquad +2\int_{I\!R^9} \sigma^r \;\; \delta_{\mathcal{E}'} f_0^e(v_e^\star) \;\; f_0^i(v_i)\Big[\phi^e({v_e}')\;\;\phi^e(v_e^\star)+\phi^i(v_i)\;\;\phi^e({v_e}')\\
&\qquad\qquad\qquad\qquad +\phi^i(v_i)\;\;\phi^e(v_e^\star)-\phi^e(v_e)\;\;\phi^n(v_i)\Big]dv_i \;\; dv_e^\star \;\; d{v_e}',\\
&(f_0^i(v_i))^{-1} DQ^{i,ir}(\phi^e,\phi^i,\phi^n)(v_i)\\
&\quad = \int_{I\!R^9} \sigma^r \;\; \delta_{\mathcal{E}} f_0^e({v_e}') \;\; f_0^e(v_e^\star)\Big[\phi^e(v_e)\;\;\phi^n(v_i)-\phi^e({v_e}')\;\;\phi^e(v_e^\star)\\
&\qquad\qquad\qquad\qquad -\phi^i(v_i)\;\;\phi^e({v_e}')-\phi^i(v_i)\;\;\phi^e(v_e^\star)\Big]dv_e \;\; dv_e^\star \;\; d{v_e}',
\end{aligned}
$$

$$\begin{aligned}
&\left(f_0^n(v_n)\right)^{-1} DQ^{n,ir}\left(\phi^e,\phi^i,\phi^n\right)(v_n)\\
&\quad=\mathcal{F}_0\int_{I\!R^9}\sigma^r\delta_{\mathcal{E}}f_0^e(v_e)\Big[\phi^e(v_e{}')\phi^e(v_e^\star)+\phi^i(v_i)\ \phi^e(v_e{}')\\
&\qquad\qquad+\phi^i(v_i)\ \phi^e(v_e^\star)-\phi^e(v_e)\ \phi^n(v_i)\Big]dv_e\ dv_e^\star\ dv_e{}',
\end{aligned}$$

where $\delta_{\mathcal{E}}$ stands for $|v_e|^2=|v_e{}'|^2+|v_e^\star|^2+2\Delta$. Finally, the remainder terms R^β (which are linked to the order two corrections in the conservation equations) are such that:

$$R^e=0,\quad \int_{I\!R^3}R^i(v_i)\,dv_i=0,\quad \int_{I\!R^3}R^n(v_n)\,dv_n=0. \tag{31}$$

Proof. We start from the definition (7a)–(7c) of the ionization-recombination operators and set: $f_0^\alpha=\rho_\alpha M_{u,T}$ for $\alpha=i,n$, and $f_0^e=\rho_e M_{0,T_e}$. We notice that the scaled conservation equations (16) give:

$$\begin{aligned}
v_n &= v_i+\mathcal{O}(\varepsilon^2),\\
|v_e|^2 &= |v_e{}'|^2+|v_e^\star|^2+2\Delta+(|v_i|^2-|v_n|^2)=|v_e{}'|^2+|v_e^\star|^2+2\Delta+\mathcal{O}(\varepsilon^2),
\end{aligned}$$

so that, thanks to (25), we deduce that:

$$\begin{aligned}
&\mathcal{F}_0 f_0^e(v_e)f_0^n(v_n)-f_0^e(v_e{}')f_0^e(v_e^\star)f_0^i(v_i)\\
&\quad=\mathcal{F}_0\ f_0^e(v_e)\ f_0^n(v_n)\left[1-exp\left(-\frac{|v_i|^2-|v_n|^2}{2}\left(\frac{1}{T}-\frac{1}{T_e}\right)\right)\right]\\
&\quad=\mathcal{F}_0\ f_0^e(v_e)\ f_0^n(v_n)\frac{|v_i|^2-|v_n|^2}{2}\left(\frac{1}{T}-\frac{1}{T_e}\right)+\mathcal{O}(\varepsilon^4)\\
&\quad=\mathcal{O}(\varepsilon^2);
\end{aligned}$$

this gives (29), and the order two correction above allows to compute the remainders R^β. Now, inserting the expansion (28) in each ionization-recombination collision term, we obtain, after some easy (but lengthy) computations, the expression of the remaining terms in (30). ∎

5. THE HYDRODYNAMIC MODEL

We start from the system of scaled kinetic equations (18), replace the elastic collision operators $Q_\varepsilon^{\alpha\beta}$ by the expansions proposed in Lemma 4.1 and 4.2, and introduce the diffusion scaling of small parameter ε: $t \to \varepsilon^2 t$, $x \to \varepsilon x$. We obtain:

$$\begin{aligned}\partial_t f_\varepsilon^e + \varepsilon^{-1}\left(v_e \cdot \nabla_x f_\varepsilon^e + F_e \cdot \nabla_{v_e} f_\varepsilon^e\right) &= \varepsilon^{-2} Q_0^{en}(f_\varepsilon^e, f_\varepsilon^n) \\ &+ \varepsilon^{-1}\Big[Q_1^{en}(f_\varepsilon^e, f_\varepsilon^n) + Q_0^{ei}(f_\varepsilon^e, f_\varepsilon^i) + Q^{ee}(f_\varepsilon^e, f_\varepsilon^e) + Q^{e,ir}(f_\varepsilon^e, f_\varepsilon^i, f_\varepsilon^n)\Big] \\ &+ Q_2^{en}(f_\varepsilon^e, f_\varepsilon^n) + Q_1^{ei}(f_\varepsilon^e, f_\varepsilon^i) + \varepsilon\Big[\ Q_3^{en}(f_\varepsilon^e, f_\varepsilon^n) + Q_2^{ei}(f_\varepsilon^e, f_\varepsilon^i)\Big] + \mathcal{O}(\varepsilon^2),\end{aligned} \tag{32}$$

$$\begin{aligned}\partial_t f_\varepsilon^i + v_i \cdot \nabla_x f_\varepsilon^i + F_i \cdot \nabla_{v_i} f_\varepsilon^i &= \varepsilon^{-1}\Big[Q_0^{in}(f_\varepsilon^i, f_\varepsilon^n) + Q^{i,ir}(f_\varepsilon^e, f_\varepsilon^i, f_\varepsilon^n)\Big] + Q^{ii}(f_\varepsilon^i, f_\varepsilon^i) + Q_0^{ie}(f_\varepsilon^i, f_\varepsilon^e) \\ &+ \varepsilon\left[\frac{1}{2}\ Q_0^{in}(f_\varepsilon^i, f_\varepsilon^n) + Q_2^{in}(f_\varepsilon^i, f_\varepsilon^n) + Q_1^{ie}(f_\varepsilon^i, f_\varepsilon^e)\right] + \mathcal{O}(\varepsilon^2),\end{aligned} \tag{33}$$

$$\begin{aligned}\partial_t f_\varepsilon^n + v_n \cdot \nabla_x f_\varepsilon^n + F_n \cdot \nabla_{v_n} f_\varepsilon^n &= \varepsilon^{-1} Q^{nn}(f_\varepsilon^n, f_\varepsilon^n) \\ &+ Q_0^{ne}(f_\varepsilon^n, f_\varepsilon^e) + Q_0^{ni}(f_\varepsilon^n, f_\varepsilon^i) + Q^{n,ir}(f_\varepsilon^e, f_\varepsilon^i, f_\varepsilon^n) \\ &+ \varepsilon\ Q_1^{ne}(f_\varepsilon^n, f_\varepsilon^e) + \mathcal{O}(\varepsilon^2).\end{aligned} \tag{34}$$

Next, we expand the solutions in powers of ε,

$$f_\varepsilon^e = f_0^e + \varepsilon\ f_1^e + \varepsilon^2\ f_2^e + \mathcal{O}(\varepsilon^3) \quad \text{and} \quad f_\varepsilon^\alpha = f_0^\alpha + \varepsilon\ f_1^\alpha + \mathcal{O}(\varepsilon^2), \quad \alpha = i, n, \tag{35}$$

setting $f_1^\beta = f_0^\beta\ \phi_1^\beta$ with $\beta = e, i, n$. Then, we insert these expansions in the system (32)–(34) and identify terms of equal powers of ε. This leads to a system of equations to be successively solved. The identification of the lowest order terms first gives the equilibrium states for each species: we refer to Theorem 5.1 below. Then the identification of the following terms in (32)–(34) allow to compute the order one corrections, denoted by f_1^α ($\alpha = e, i, n$), as expressed in Lemmas 5.2 ($\alpha = n$), 5.3 ($\alpha = e$) and 5.4 ($\alpha = i$). Solvability conditions finally give the coupled fluid model for the mixture, which is stated in Theorem 5.6 below. We now detail these different steps. For clarity, the proofs of the forthcoming statements are postponed to the next section.

Let us start with the identification of the lowest order terms, which are of order ε^{-2} for the electrons (in Eq. (32)) and ε^{-1} for the other species, *i.e.* in Eqs. (33), (34). We get:

$$Q_0^{en}(f_0^e, f_0^n)(v_e) = 0, \tag{36}$$

$$Q_0^{in}\left(f_0^i, f_0^n\right)(v_i) + Q_0^{i,ir}(f_0^e, f_0^i, f_0^n)(v_i) = 0, \tag{37}$$

$$Q_0^{nn}\left(f_0^n, f_0^n\right)(v_n) = 0. \tag{38}$$

This allows to derive the equilibrium distribution functions f_0^n, f_0^i and f_0^e. More precisely, we obtain:

Theorem 5.1. The equilibrium distribution functions of neutral particles f_0^n and ions f_0^i are Maxwellians characterized by the same mean velocity u and temperature T:

$$f_0^\alpha(v_\alpha) = \rho_\alpha M_{\mathrm{u},T}(v_\alpha) = \frac{\rho_\alpha}{(2\pi T)^{3/2}} \exp\left(-\frac{|v_\alpha - \mathrm{u}|^2}{2T}\right), \quad \text{with} \quad \alpha = i, n. \tag{39}$$

The equilibrium distribution function of electrons is the centered Maxwellian of temperature T_e:

$$f_0^e(v_e) = \rho_e M_{0,T_e}(v_e) = \frac{\rho_e}{(2\pi T_e)^{3/2}} \exp\left(-\frac{|v_e|^2}{2T_e}\right), \tag{40}$$

where the electron density ρ_e is governed by the ion density ρ_i, the density of neutral particles ρ_n and the electron temperature T_e, according to

$$\rho_e = \frac{\mathcal{F}_0\ \rho_n}{\rho_i}(2\pi T_e)^{3/2} \exp\left(-\frac{\Delta}{T_e}\right). \tag{41}$$

Remark. This closure equation for defining ρ_e is a Saha law generalized to a plasma characterized by two temperatures: a temperature for heavy particles and another temperature for light particles. To our knowledge, there exist two distinct generalizations of the Saha law for two temperature plasma: Potapov's equation,[20,21] and Eindhoven's equation.[22–24] They differ due to the definition used for the internal partition functions, Z_i, Z_n, which is generalized for the former and not for the latter. The generalized Saha law we obtain Eq. (41) turns out to be a dimensionless version of Eindhoven's formulation

$$\rho_e = 2\ \frac{Z_i}{Z_n}\ \frac{\rho_n}{\rho_i}\ \left(\frac{2\pi m_e k T_e}{h^2}\right)^{3/2}\ \exp\left(-\frac{\Delta}{kT_e}\right), \tag{42}$$

where h denotes Planck constant and the ionization lowering is neglected (since we assumed Δ constant). The kinetic derivation proposed in this study thus supports previous argumentations, such as in ref. 25 and references within, in favour of Eindhoven's generalization of the Saha law. The other formulation does not seem to be consistent with the entropy inequality (which is needed to derive the macroscopic limit).

Then we introduce the conservative variable $\mathbf{u} = (\rho_n, \rho_i, \rho_n u, \rho_n E, \rho_e E_e)$: $I\!R^3 \times I\!R^+ \to \mathcal{U}$ where $\mathcal{U}$ denotes the open set $\{\mathbf{u} \in I\!R^7, \rho_n \in I\!R^+, \rho_i \in I\!R^+, \rho_n u \in I\!R^3, T \in I\!R^+, T_e \in I\!R^+, \}$. The density ρ_n, the momentum $\rho_n u$ and the energy $\rho_n E = \rho_n(|u|^2 + 3T)/2$ of neutral particles are such that:

$$\begin{pmatrix} \rho_n \\ \rho_n u \\ \rho_n E \end{pmatrix} = \int_{I\!R^3} \begin{pmatrix} 1 \\ v_n \\ \frac{1}{2}|v_n|^2 \end{pmatrix} f_0^n(v_n) dv_n. \tag{43}$$

The ion density ρ_i and the electron internal energy $\rho_e E_e = \frac{3}{2}\rho_e T_e$ are defined by

$$\rho_i = \int_{I\!R^3} f_0^i(v_i)\ dv_i \quad \text{and} \quad \rho_e E_e = \frac{1}{2}\int_{I\!R^3} |v_e|^2\ f_0^e(v_e) dv_e. \tag{44}$$

Equipped with these notations, we now go on and identify terms of order ε^{-1} in the kinetic equation associated with the electrons, and in the same way, constant terms in the Eqs. (33)–(34). Thanks to Lemma 4.6, we have in particular $Q_0^{e,ir}\left(f_0^e, f_0^i, f_0^n\right) = Q_0^{n,ir}\left(f_0^e, f_0^i, f_0^n\right) = 0$, so that we get:

$$-v_e\cdot\nabla_x f_0^e(v_e)-F_e\cdot\nabla_{v_e} f_0^e(v_e)+Q_0^{en}\left(f_1^e,f_0^n\right)(v_e)+Q_0^{en}\left(f_0^e,f_1^n\right)(v_e)$$
$$+Q_1^{en}\left(f_0^e,f_0^n\right)(v_e)+Q^{ee}\left(f_0^e,f_0^e\right)(v_e)+\ Q_0^{ei}\left(f_0^e,f_0^i\right)(v_e)=0, \tag{45}$$
$$-\partial_t f_0^i(v_i)-v_i\cdot\nabla_x f_0^i(v_i)-F_i\cdot\nabla_{v_i} f_0^i(v_i)+Q_0^{in}(f_1^i,f_0^n)(v_i)+Q_0^{in}(f_0^i,f_1^n)(v_i)$$
$$+\mathcal{L}Q^{i,ir}(\phi_1^e,\phi_1^i,\phi_1^n)(v_i)+Q_0^{ie}(f_0^i,f_0^e)(v_i)+Q^{ii}(f_0^i,f_0^i)(v_i)=0, \tag{46}$$
$$-\partial_t f_0^n(v_n)-v_n\cdot\nabla_x f_0^n(v_n)-F_n\cdot\nabla_{v_n} f_0^n(v_n)+2\ Q^{nn}\left(f_1^n,f_0^n\right)(v_n)$$
$$+Q_0^{ne}\left(f_0^n,f_0^e\right)(v_n)+Q_0^{ni}\left(f_0^n,f_0^i\right)(v_n)=0. \tag{47}$$

This identification allows to determine the first order corrective terms $f_1^\alpha(v_\alpha)$, with $\alpha=e,n,i$; moreover, solvability conditions appear in this computation, which lead to a fluid model for neutrals (Lemma 5.2) and ions (Lemma 5.4).

Let us start with the easiest case, that means with neutrals. The linear operator associated with this species is in fact the classical mono-species linearized Boltzmann operator, here denoted by L_{nn}, and defined ref. 26 by:

$$L_{nn}\phi=2\rho_n M_{u,T}^{-1}\ Q^{nn}\left(M_{u,T},M_{u,T}\phi\right). \tag{48}$$

This operator also satisfies:

$$L_{nn}\phi(v_n)=\rho_n\int_{I\!R^3\times S^2} B_\star^{\mathcal{B}}(v_n-v_n^\star,\Omega)\ M_{u,T}(v_n^\star)$$
$$\times\left[\phi(v_n^{\star\prime})+\phi(v_n{}')-\phi(v_n^\star)-\phi(v_n)\right]dv_n^\star\ d\Omega,$$

and classical Boltzmann theory[26] allows to derive the following result:

Lemma 5.2. The solution f_1^n of Eq. (47) exists if and only if the density ρ_n of neutral particles, their velocity u and their temperature T are governed by the following fluid system ($t>0,\ \ x\in I\!R^3$):

$$\begin{aligned}&\partial_t\rho_n+\operatorname{div}(\rho_n u)=0,\\&\partial_t(\rho_n u)+\operatorname{div}[\rho_n(u\otimes u)]+\nabla_x(\rho_n T)-\rho_n F_n=0,\\&\partial_t(\rho_n E)+\operatorname{div}[\rho_n u(E+T)]-\rho_n u\cdot F_n=0,\end{aligned} \tag{49}$$

where $E=\frac{1}{2}|u|^2+\frac{3}{2}T$.

Assuming that (49) is verified, there exists a unique solution $f_1^n = f_0^n \phi_1^n$ of Eq. (47) with ϕ_1^n in (Ker $L_{nn})^\perp$, so that in particular, we have:

$$\int_{\mathbb{R}^3} \phi_1^n(v)\, M_{u,T}(v)\, dv = 0. \tag{50}$$

It is given by:

$$\phi_1^n(v_n) = -\frac{1}{\rho_n}\left[a_n(T,|V_n|)\;A(V_n)\cdot\frac{\nabla_x T}{\sqrt{T}} + \frac{1}{2}\;b_n(T,|V_n|)\;B(V_n):\sigma(u)\right], \tag{51}$$

with the notations:

$$V_n = \frac{v_n - u}{\sqrt{T}} \quad \text{and} \quad A(v) = \left(\frac{1}{2}\;|v|^2 - \frac{5}{2}\right)\; v;$$

moreover, the traceless tensors B and σ (which is the strain tensor) are defined by

$$B(v) = v \otimes v - \frac{1}{3}\;|v|^2\;\mathbf{Id} \quad \text{and} \quad \sigma_{ij}(v) = \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} - \frac{2}{3}\;\mathrm{div}(v)\;\delta_{ij}.$$

Finally, the scalar functions $a_n(T,|V_n|)$ and $b_n(T,|V_n|)$ are such that

$$A'(V_n) = -\frac{1}{\rho_n} a_n(T,|V_n|)\;A(V_n) \quad \text{and} \quad B'(V_n) = -\frac{1}{\rho_n} b_n(T,|V_n|)\;B(V_n)$$

are the unique solutions in (Ker $L_{nn})^\perp$ of equations $L_{nn}A' = A$ and $L_{nn}B' = B$.

The next lemma is devoted to the computation of $f_1^e = f_0^e \phi_1^e$, according to Eq. (45), and uses the properties of the linear operator L_{en} studied in Lemma 4.3. As we will see below, no solvability condition appears during this computation, because the right hand side of the equation satisfied by ϕ_1^e is odd. It is the reason why, we will have to go on in this identification (just for the electrons) and consider the solvability condition of the equation satisfied by the order two correction $f_2^e = f_0^e \phi_2^e$, in order to derive a fluid model for this species. For the moment, we have the following explicit computation (this is specific to the case of Lorentz operators) of the first order correction f_1^e:

Lemma 5.3. The unique solution $f_1^e = f_0^e \phi_1^e$ of Eq. (45) with ϕ_1^e in $(\mathrm{Ker}\ L_{en})^{\perp}$, is given by

$$\begin{aligned} \phi_1^e(v_e) &= v_e \cdot \bar{\phi}_1^e(v_e), \\ \bar{\phi}_1^e(v_e) &= \frac{u}{T_e} - \frac{1}{2\alpha(v_e)\rho_n}\left[\frac{\nabla_x \rho_n}{\rho_n} - \frac{\nabla_x \rho_i}{\rho_i} - \frac{F^e}{T_e} + \frac{\nabla_x T_e}{T_e}\left(\frac{|v_e|^2 + 2\Delta}{2T_e}\right)\right], \end{aligned} \tag{52}$$

where α is the isotropic function defined by

$$\alpha(v) = \alpha(|v|) = \int_{S_+^2} B(v, \Omega)\frac{(v, \Omega)^2}{|v|^2}\, d\Omega.$$

We end with the computation of $f_1^i = f_0^i \phi_1^i$ for the ions. Among the order one corrections, this is the newest computation. It is based on the properties of the linear operator L_{in} studied in Lemma 4.4. This leads to only one solvability condition, which gives a fluid equation for the density of the ions; the macroscopic behaviour of the velocity u and the temperature T, which are common with the neutrals, has been in fact already determined (see Lemma 5.2 above). Using all the notations of the preceding lemmas, we have:

Lemma 5.4. If the solution $f_1^i = f_0^i \phi_1^i$ of equation (46) exists, with ϕ_1^i in $(\mathrm{Ker}\ L_{in})^{\perp}$, *i.e.* such that:

$$\int_{IR^3} \phi_1^i(v)\, M_{u,T}(v)\, dv = 0, \tag{53}$$

then the density ρ_i of the ions has to satisfy the following fluid equation:

$$\partial_t \rho_i + \mathrm{div}(\rho_i\ u) = \rho_i\ \mathcal{Y}^{ir}, \tag{54}$$

where $\mathcal{Y}^{ir}$ is defined by ($\delta_{\mathcal{E}}$ stands for $|v_e|^2 = |v_e'|^2 + |v_e^\star|^2 + 2\Delta$):

$$\mathcal{Y}^{ir} = \int_{IR^9} \sigma^r\ \delta_{\mathcal{E}} f_0^e(v_e')\ f_0^e(v_e^\star)\left[\phi_1^e(v_e) - \phi_1^e(v_e') - \phi_1^e(v_e^\star)\right] dv_e\ dv_e'\ dv_e^\star. \tag{55}$$

Conversely, let us suppose that conditions (54)–(55) are fulfilled. Then, Eq. (46) has a unique solution $f_1^i = f_0^i \phi_1^i$ with (53); moreover, setting $V_i = (v_i - u)T^{-1/2}$, ϕ_1^i is given by

$$\phi_1^i(v_i) = \phi_1^n(v_i) \; - \; c_i \, V_i \cdot \left(\frac{\nabla_x \rho_i}{\rho_i} - \frac{\nabla_x \rho_n}{\rho_n} + \frac{F_n - F_i}{T} \right), \tag{56}$$

where the scalar function $c_i = c_i(\rho_i, \rho_n, |V_i|, T, T_e)$ is isotropic with respect to V_i.

As we have seen before, in order to derive a fluid model for the electrons, we have to continue and identify terms of order ε^0 in the kinetic equation (32). This gives

$$\begin{aligned}
&-\partial_t f_0^e(v_e) - v_e \cdot \nabla_x f_1^e(v_e) - F_e \cdot \nabla_{v_e} f_1^e(v_e) \\
&\quad + Q_0^{en}\left(f_2^e, f_0^n\right)(v_e) + Q_0^{en}\left(f_0^e, f_2^n\right)(v_e) \\
&\quad + Q_0^{en}\left(f_1^e, f_1^n\right)(v_e) + Q_1^{en}\left(f_1^e, f_0^n\right)(v_e) \\
&\quad + Q_1^{en}\left(f_0^e, f_1^n\right)(v_e) + 2Q^{ee}\left(f_1^e, f_0^e\right)(v_e) \\
&\quad + Q_0^{ei}\left(f_1^e, f_0^i\right)(v_e) + Q_0^{ei}\left(f_0^e, f_1^i\right)(v_e) + \mathcal{L}Q^{e,ir}\left(\phi_1^e, \phi_1^i, \phi_1^n\right)(v_e) \\
&\quad + Q_2^{en}\left(f_0^e, f_0^n\right)(v_e) + Q_1^{ei}\left(f_0^e, f_0^i\right)(v_e) = 0.
\end{aligned} \tag{57}$$

Then, establishing the solvability condition for this equation, we get:

Lemma 5.5. If the solution f_2^e of Eq. (57) exists, then the electron temperature T_e satisfies

$$\begin{aligned}
&\partial_t T_e + u \cdot \nabla_x T_e + \frac{2T_e^2}{3T_e + 2\Delta} \left(\operatorname{div}(u) + \frac{1}{\rho_e} \operatorname{div}(\rho_e u_J) \right) \\
&\quad = \left(1 + \frac{\rho_i}{\rho_e} \right) \; \frac{2T_e^2}{3T_e + 2\Delta} \mathcal{Y}^{ir},
\end{aligned} \tag{58}$$

where we have set:

$$u_J = -d_1 \left(\frac{\nabla_x \rho_n}{\rho_n} - \frac{\nabla_x \rho_i}{\rho_i} - \frac{F^e}{T_e} \right) - (d_2 + \Delta d_1) \frac{\nabla_x T_e}{(T_e)^2}, \tag{59}$$

with:

$$d_1 = \frac{1}{6\rho_n} \int_{I\!R^3} \frac{|v|^2}{\alpha(|v|)} M_{0,T_e}(v) dv, \quad d_2 = \frac{1}{12\rho_n} \int_{I\!R^3} \frac{|v|^4}{\alpha(|v|)} M_{0,T_e}(v) dv. \tag{60}$$

Finally, gathering all the previous results, we obtain the following hydrodynamic/diffusion system governing the time evolution of the neutral and ion densities ρ_n and ρ_i, the velocity u and the temperature T of the heavy particles, and the electron temperature T_e.

Theorem 5.6. The inviscid hydrodynamic/diffusion system derived from Eq. (32)–(34) is:

$$
\begin{aligned}
&\partial_t \rho_n + \operatorname{div}(\rho_n u) = 0, \quad t > 0, \ x \in I\!R^3,\\
&\partial_t \rho_i + \operatorname{div}(\rho_i u) = \rho_i \ \mathcal{Y}^{ir},\\
&\partial_t (\rho_n u) + \operatorname{div}[\rho_n (u \otimes u)] + \nabla_x (\rho_n T) - \rho_n F_n = 0,\\
&\partial_t (\rho_n E) + \operatorname{div}[\rho_n u (E + T)] - \rho_n u \cdot F_n = 0,\\
&\partial_t T_e + u \cdot \nabla_x T_e + \frac{2T_e^2}{3T_e + 2\Delta} \left(\operatorname{div}(u) + \frac{1}{\rho_e} \operatorname{div}(\rho_e u_J) \right)\\
&\quad = \left(1 + \frac{\rho_i}{\rho_e}\right) \ \frac{2T_e^2}{3T_e + 2\Delta} \mathcal{Y}^{ir},
\end{aligned}
$$

with the energy $E = \frac{1}{2} \ |u|^2 + \frac{3}{2} \ T$. The ionization-recombination source term $\mathcal{Y}^{ir}$ is defined in Lemma 5.4 and the diffusion velocity u_J is given by (59)–(60).
This system is supplemented by the closure relation (41).

Remark 1. This fluid limit, which is valid for both the pre-sheath and plasma column of an arc discharge, is inviscid. The viscous case is the object of a forthcoming study.[14] But differences already appear compared to previous models mentioned in the introduction. Let us do some comments and comparisons.

We start discussing kinetic approaches, with Devoto's model.[7] This author does two main assumptions to derive a simplified model describing the transport properties of ionized gas mixtures. These simplifications allow avoiding the expensive computations obtained with the exact theory. First, the electron-heavy collision terms are neglected in deriving expressions for the ion and atom transport properties. Secondly, the change in the heavy perturbation term during a collision is also neglected in obtaining expressions for the electron transport properties. This leads to an hydrodynamic model for the heavy species which does not depend on the electrons. Conversely, the heavy species appear in the computation of the transport coefficients for the electrons (electron diffusive flux, electron thermal flux). Our model is different for essentially two points. First, we consider weakly ionized plasma ($\delta \approx 10^{-2}$), so that the leading order collisional term for the electrons is only due to elastic collisions with neutrals. As a consequence, we can compute explicitly the order one correction f_1^e, and the diffusion coefficients too. This is not the case in ref. 7, where the collisions with electrons (and ions too) are also of the same order of magnitude (which would correspond to a highly ionized plasma where $\delta \approx 1$). This explains in particular the expansion in a series of Sonine polynomials we can find in ref. 7, and also the discussion concerning the degree of its truncation (in order to get a sufficient accuracy in the computation of the transport coefficients). Secondly, it seems that only elastic collisions have been taken into account in ref. 7. The inelastic collisions play in fact a very significant role in our study, both at the kinetic and the fluid level. They govern in particular: (i) the production of mass for the ions (see Eq. (54)), (ii) also the equilibrium state for the electrons; f_0^e would be only an isotropic function otherwise. Here it is a Maxwellian, although the intra species collision term Q^{ee} is not dominant. (iii) And also, naturally, the Saha law which does not exist in ref. 7 and which allows here reducing the fluid model for the electrons to only one equation on their temperature.

The arc discharge model (for the cathode region) of ref. 1 is developed from the kinetic scale using physical arguments. Electrons and ions are governed by Devoto's model discussed above, but not neutrals. Their density is indeed governed by Potapov's generalization of the Saha law to account for ionization and recombination. As already underlined, the frame of assumptions introduced in refs. 7 and 1 to derive fluid limits for electrons and heavy species differ, while a common frame is used here for all the species. An other difference, discussed in the remark following Theorem 5.1, is the formulation of the generalized Saha law; the present study indeed leads to the other formulation, namely Eindhoven's equation. Moreover, the present work also applies to unsteady situations with mass flow, contrary to ref. 1.

As arc discharges can involve unsteady and three-dimensional effects, macroscopic models with these features have also been used in refs. 3 and 6 for instance. The drawback of these fluid approaches is that the pre-sheath is modelled as the plasma column. Consequently ionization is neglected, local thermal equilibrium is assumed not only for each species but also between the species (one temperature model).

Remark 2. Let us conclude with some comment about Assumption 2.1. What happens if there is a magnetic field B? In fact, most of the results are conserved, or slightly modified. In particular, the hydrodynamic model derived in Theorem 5.6 is the same, apart from the fact that the diffusion coefficients d_i $(i=1,2)$, which appear in (59), are now matrices with an antisymmetric part, due to the presence of the magnetic field. We refer to a connected work [17] for the details in the computations.

6. PROOFS

We now turn proving our two main results, which are: the equilibrium states and the "Saha" law (Theorem 5.1), and the fluid model for the whole system (Theorem 5.6).

Proof of Theorem 5.1. The equilibrium distribution function f_0^n associated with the neutral particles is determined from Eq. (38). We first notice that, similarly to the partially ionized plasma studied in ref. 11, f_0^n is not influenced by electrons and ions. Applying the classical theory of the Boltzmann equation to (38), we can immediately conclude that there exists $u \in I\!R^3$, ρ_n and $T \in I\!R^+$ such that:

$$f_0^n(v_n) = \rho_n M_{u,T}(v_n), \tag{61}$$

which gives (39) for $\alpha = n$.
Referring to Lemma 4.1 and 4.3, Eq. (45) means that f_0^e has to be isotropic, i.e. $f_0^e(v_e) = f_0^e(|v_e|)$.
Consequently, Eq. (37) reads:

$$Q_0^{in}\left[f_0^i, \rho_n M_{u,T}\right] + Q^{i,ir}\left[f_0^e, f_0^i, \rho_n M_{u,T}\right] = 0. \tag{62}$$

We now turn specifying the solution f_0^i of (62) in the form $f_0^i = M_{u,T}\ \phi_0^i$, with ϕ_0^i to be determined. Let us set:

$$\rho_i = \int_{I\!R^3} f_0^i(v_i)\ dv_i = \int_{I\!R^3} M_{u,T}(v_i)\ \phi_0^i(v_i)\ dv_i.$$

Using the definition (7b) of $Q^{i,ir}$ and $v_n = v_i$ (which is the limit (24) of δ_v when $\varepsilon \to 0$), we obtain:

$$Q^{i,ir}\left(f_0^e, f_0^i, \rho_n M_{u,T}\right)(v_i) = \left[\mathcal{A}_1 - \mathcal{A}_2\ \rho_i^{-1}\ \phi_0^i(v_i)\right]\ M_{u,T}(v_i), \tag{63}$$

where $\mathcal{A}_1$ and $\mathcal{A}_2$ denote the following positive constants (thanks to (8), $\mathcal{A}_1$ and $\mathcal{A}_2$ are in fact independent of v_i):

$$\begin{aligned}\mathcal{A}_1 &= \rho_n \int_{IR^9} \sigma^r(v_e', v_e^\star; v_e)\ \delta_{\mathcal{E}}\ \mathcal{F}_0\ f_0^e(|v_e|)\ dv_e\ dv_e'\ dv_e^\star,\\ \mathcal{A}_2 &= \rho_i \int_{IR^9} \sigma^r(v_e', v_e^\star; v_e)\ \delta_{\mathcal{E}}\ f_0^e(|v_e'|)\ f_0^e(|v_e^\star|)\ dv_e\ dv_e'\ dv_e^\star.\end{aligned}$$

Referring to the definition (23) of L_{in}, the determination of f_0^i solution of (62) reduces then to the derivation of the positive function ϕ_0^i solution of

$$L_{in}\phi_0^i = S_0^i, \qquad \text{where} \qquad S_0^i(v_i) = -\Big[\mathcal{A}_1 - \mathcal{A}_2\ \rho_i^{-1}\ \phi_0^i(v_i)\Big]. \tag{64}$$

As the right hand side S_0^i here depends on the unknown function ϕ_0^i, the solvability condition given by Lemma 4.4 only appears as a necessary condition. So, if ϕ_0^i exists, we must have $\int_{IR^3} S_0^i M_{u,T}\, dv = 0$, or equivalently:

$$\mathcal{A}_1 = \mathcal{A}_2.$$

We simply denote by $\mathcal{A}$ this common value, i.e. $\mathcal{A}_1 = \mathcal{A}_2 = \mathcal{A}$.
Conversely, let us now assume that this condition $\mathcal{A}_1 = \mathcal{A}_2 = \mathcal{A}$ is satisfied; we want to show that Eq. (64) has a solution. We first remark that the constant function ϕ_0^i defined by $\phi_0^i(v_i) = \rho_i$ is clearly a solution of (64), and that it is the only constant solution of (64). To prove that this solution is unique, let $\varphi = \phi_0^i - \phi$, where ϕ is supposed to be another positive solution of (64). Then φ belongs to the kernel of $(L_{in} - \mathcal{A}\rho_i^{-1} Id)$. Now, there are two cases:

(i) If $\mathcal{A} = 0$, then, by Lemma 4.4, $\varphi \in$ Ker L_{in} is constant and ϕ also. But since Eq. (64) admits at most one constant solution, we deduce that $\phi = \phi_0^i$.

(ii) If $\mathcal{A} \neq 0$. As $(L_{in} - \mathcal{A}\rho_i^{-1} Id)\varphi = 0$, we obtain:

$$\begin{aligned}&\int_{IR^3} \Big(L_{in} - \mathcal{A}\rho_i^{-1}\Big)(\varphi)\ \varphi\ M_{u,T}\ dv_i = 0\\ &\quad = \int_{IR^3} L_{in}\varphi\ \varphi\ M_{u,T}\ dv_i - \mathcal{A}\ \rho_i^{-1} \int_{IR^3} \varphi^2\ M_{u,T}\ dv_i.\end{aligned}$$

We know from the weak formulation (cf. proof of Lemma 4.4) that the first integral on the right hand side is negative. But as $\mathcal{A}\rho_i^{-1}$ is positive, we deduce that each term on the right hand side has to be zero, which gives $\varphi = 0$ and $\phi = \phi_0^i$.

Consequently, the equilibrium distribution function for ions, f_0^i, is the Maxwellian characterized by the same mean velocity u and temperature T than the neutral particles:

$$f_0^i = \rho_i M_{u,T}, \tag{65}$$

provided that $\mathcal{A}_1 = \mathcal{A}_2$.
Let us now look more precisely at this condition. As $\phi_0^i = \rho^i$, Eq. (63) shows that, under the assumption $\mathcal{A}_1 = \mathcal{A}_2$, we have: $Q^{i,ir}(f_0^e, f_0^i, f_0^n) = 0$. In other words, there is no production or annihilation of ions due to ionization-recombination reactions at the order $\mathcal{O}(\varepsilon^{-1})$. This means naturally that, still at this order, there is no production or annihilation of electrons due to ionization-recombination, so that: $Q^{e,ir}(f_0^e, f_0^i, f_0^n) = 0$ too. From the point (iii) of Lemma 4.5, we deduce that the isotropic function f_0^e is such that: $f_0^e = \rho_e M_{0,T_e}$, with ρ_e given by (41), which concludes the proof. ∎

Proof of Lemma 5.2. Setting $f_1^n(v_n) = f_0^n(v_n)\phi_1^n(v_n)$, we derive the first corrective term f_1^n from Eq. (47). We first notice that the collision terms $Q_0^{ni}\left(f_0^n, f_0^i\right)$ and $Q^{n,ir}\left(f_0^e, f_0^i, f_0^n\right)$ are equal to zero. A similar result holds for $Q_0^{ne}\left(f_0^n, f_0^e\right)$, since f_0^e and $B^{\mathcal{B}}$ are even functions of v_e. Then (47) reduces to

$$L_{nn}\phi_1^n = \left(f_0^n\right)^{-1}\left[\partial_t\, f_0^n + v_n \cdot \nabla_x f_0^n + F_n \cdot \nabla_{v_n} f_0^n\right], \tag{66}$$

where L_{nn} denotes the classical linearized Boltzmann operator defined by (48). The solvability condition of the equation $L_{nn}\phi_1^n = \varphi$ is a well known result (cf. ref. 26 for instance):

$$\int_{I\!R^3} \varphi(v_n) \begin{pmatrix} 1 \\ v_n \\ \frac{1}{2}|v_n|^2 \end{pmatrix} f_0^n(v_n)\; dv_n = 0. \tag{67}$$

When applied to the present problem, this condition exactly leads to (49). Setting $V_n = (v_n - u)T^{-1/2}$ and using (49) to express the time derivatives of ρ_n, u and T, the right hand side of (66) can be written

$$\Big[\partial_t + v_n \cdot \nabla_x + F_n \cdot \nabla_{v_n}\Big] f_0^n(v_n) = f_0^n(v_n)\ \left(\frac{1}{2}B(V_n) : \sigma(u) + A(V_n) \cdot \frac{\nabla_x T}{\sqrt{T}}\right). \tag{68}$$

The vector A, the tensor B and the traceless rate of strain tensor σ are detailed in Lemma 5.2. We know from ref. 27 that

$$L_{nn} A'(V_n) = A(V_n) \quad \text{and} \quad L_{nn} B'(V_n) = B(V_n) \tag{69}$$

have a unique solution in (Ker L_{nn})$^{\perp}$ of the form

$$A'(V_n) = -\frac{1}{\rho_n} a_n(T, |V_n|)\ V_n \quad \text{and} \quad B'(V_n) = -\frac{1}{\rho_n} b_n(T, |V_n|)\ B(V_n),$$

where $a_n(T, |V_n|)$ and $b_n(T, |V_n|)$ are scalar functions satisfying the orthogonality requirements. Thus the expression of ϕ_1^n. ∎

Proof of Lemma 5.3. We determine the first corrective term f_1^e from Eq. (45), setting $f_1^e(v_e) = f_0^e(v_e)\phi_1^e(v_e)$. As the collisional terms $Q_0^{en}(f_0^e, f_1^n)$, $Q^{ee}(f_0^e, f_0^e)$, $Q_0^{ei}(f_0^e, f_0^i)$ and $Q^{e,ir}(f_0^e, f_0^i, f_0^n)$ are all equal to zero, Eq. (45) reads

$$L_{en}\phi_1^e = (f_0^e)^{-1}\Big[v_e \cdot \nabla_x f_0^e + F_e \cdot \nabla_{v_e} f_0^e - Q_1^{en}(f_0^e, f_0^n)\Big], \tag{70}$$

where L_{en} is defined in (20). Replacing f_0^e by the expression given in Theorem 5.1, Eq. (70) can be equivalently written:

$$L_{en}\phi_1^e = \left(\frac{\nabla_x \rho_n}{\rho_n} - \frac{\nabla_x \rho_i}{\rho_i} + \frac{|v_e|^2 + 2\Delta}{2T_e}\ \frac{\nabla_x T_e}{T_e} - \frac{F_e}{T_e}\right) \cdot v_e - (f_0^e)^{-1} Q_1^{en}(f_0^e, f_0^n).$$

As f_0^e is an even function of the velocity, we know from Lemma 4.1 (iii) that $Q_1^{en}(f_0^e, f_0^n)$ is an odd function of v_e. The solvability condition of Eq. (70) is thus satisfied, since the right hand side of the above equation is an odd function of the velocity variable v_e (cf. Lemma 4.3). Referring to the computations done in ref. 17 (our operator L_{en} is a Lorentz operator, such as in ref. 17), we can compute ϕ_1^e. We first remark that [10]

$$L_{en}(v) = -2\alpha(|v|)\rho_n v, \tag{71}$$

with α given by

$$\alpha = \alpha(|v|) = \int_{S^2_+} B(v,\Omega) \frac{(v,\Omega)^2}{|v|^2}\, d\Omega,$$

and that we also have:

$$-(f_0^e)^{-1} Q_1^{en}(f_0^e, f_0^n) = L_{en}\left(\frac{u\cdot v_e}{T_e}\right).$$

It thus remains to solve

$$L_{en}\left(\phi_1^e - \frac{u\cdot v_e}{T_e}\right) = \left(\frac{\nabla_x \rho_n}{\rho_n} - \frac{\nabla_x \rho_i}{\rho_i} + \frac{|v_e|^2 + 2\Delta}{2T_e}\ \frac{\nabla_x T_e}{T_e} - \frac{F_e}{T_e}\right)\cdot v_e,$$

which finally gives (52). ∎

Proof of Lemma 5.4. We determine the first corrective term f_1^i from Eq. (46). We write it in the form $f_1^i = f_0^i\ \phi_1^i$, with ϕ_1^i to be determined. Referring to Lemma 4.6, we first notice that $\mathcal{L}Q^{i,ir}$ can be written (using the notations of Lemma 5.4):

$$\mathcal{L}Q^{i,ir}\left(\phi_1^e, \phi_1^n, \phi_1^i\right) = f_0^i\,[\mathcal{S}_e + \mathcal{S}_{in}\ (\phi_1^n - \phi_1^i)],$$

where we have set for simplicity ($\delta_{\mathcal{E}}$ stands for $|v_e|^2 = |v_e{}'|^2 + |v_e^\star|^2 + 2\Delta$):

$$\mathcal{S}_e = \int_{I\!R^9} \sigma^r\ \delta_{\mathcal{E}}\ f_0^e(v_e{}')\ f_0^e(v_e^\star)\ \left[\phi_1^e(v_e) - \phi_1^e(v_e{}') - \phi_1^e(v_e^\star)\right]\, dv_e\ dv_e{}'\ dv_e^\star,$$

and:

$$\mathcal{S}_{in} = \int_{I\!R^9} \sigma^r\ \delta_{\mathcal{E}}\ f_0^e(v_e{}')\ f_0^e(v_e^\star)\ dv_e\ dv_e{}'\ dv_e^\star.$$

Besides, we have $Q^{ii}(f_0^i, f_0^i) = 0$ and $Q_0^{ie}(f_0^i, f_0^e) = 0$. Equation (46) can thus be written:

$$L_{in}\phi_1^i + \left(f_0^i\right)^{-1}\ Q_0^{in}\left(f_0^i, f_1^n\right) = -[\mathcal{S}_e + \mathcal{S}_{in}\ (\phi_1^n - \phi_1^i)] + \left(f_0^i\right)^{-1}\ \left[\partial_t + v_i\cdot\nabla_x + F_i\cdot\nabla_{v_i}\right] f_0^i. \tag{72}$$

From Lemma 4.4, straightforward computations show that, thanks to the fact that the operator Q_0^{in} is conservative (i.e. of zero mean value), the solvability condition for this equation, of unknown ϕ_1^i, leads to Eq. (54) with

$$\mathcal{Y}^{ir} = \frac{1}{\rho_i} \int_{I\!R^3} \mathcal{L}Q^{i,ir}\left(\phi_1^e, \phi_1^n, \phi_1^i\right)(v_i)dv_i = S_e,$$

on account of the orthogonality relations (50) and (53), which gives (55). Assume now that (54)–(55) is satisfied. To simplify the right hand side of Eq. (72), we first replace f_0^i by $\rho^i M_{u,T}$ (cf. Theorem 5.1). Next, using (49) and (54), we express the time derivatives of ρ_n, ρ_i, u and T as functions of space derivatives. We get:

$$\left(f_0^i(v_i)\right)^{-1}\left[\partial_t + v_i \cdot \nabla_x + F_i \cdot \nabla_{v_i}\right] f_0^i(v_i) = \mathcal{Y}^{ir} + (v_i - u) \cdot \Upsilon \\ + \frac{1}{2} B(V_i) : \sigma(u) + A(V_i) \cdot \frac{\nabla_x T}{\sqrt{T}},$$

with the notations (the vector A and the traceless tensors B and σ have been already defined in Lemma 5.2):

$$V_i = \frac{v_i - u}{\sqrt{T}} \quad \text{and} \quad \Upsilon = \frac{\nabla_x \rho_i}{\rho_i} - \frac{\nabla_x \rho_n}{\rho_n} + \frac{F_n - F_i}{T} .$$

Finally, as $\mathcal{Y}^{ir} = S_e$, we obtain (still using the notations $v_i{}' = v_i - (v_i - v_n, \Omega)\Omega$ and $v_n{}' = v_n + (v_i - v_n, \Omega)\Omega$):

$$\begin{aligned} &\left[L_{in}\phi_1^i + \left(f_0^i\right)^{-1} Q_0^{in}\left(f_0^i, f_1^n\right)\right](v_i) \\ &= \int_{I\!R^3 \times S^2} B_\star^{\mathcal{B}}(v_i - v_n, \Omega)\ f_0^n(v_n) \\ &\quad \times \left[\phi_1^i(v_i{}') + \phi_1^n(v_n{}') - \phi_1^i(v_i) - \phi_1^n(v_n)\right]\ dv_n\ d\Omega \\ &= (v_i - u) \cdot \Upsilon + \frac{1}{2} B(V_i) : \sigma(u) + A(V_i) \cdot \frac{\nabla_x T}{\sqrt{T}} + \mathcal{S}_{in}\ [\phi_1^i(v_i) - \phi_1^n(v_i)]. \end{aligned} \tag{73}$$

Let us set for simplicity $\psi = \phi_1^i - \phi_1^n$. As ϕ_1^n satisfies

$$\begin{aligned} L_{nn}\phi_1^n(v_i) &= \int_{I\!R^3\times S^2} B_\star^{\mathcal{B}}(v_i - v_n, \Omega)\ f_0^n(v_n) \\ &\quad \times \left[\phi_1^n(v_i{}') + \phi_1^n(v_n{}') - \phi_1^n(v_i) - \phi_1^n(v_n)\right]\ dv_n\ d\Omega \\ &= \frac{1}{2}\ B(V_i) : \sigma(u) + A(V_i)\cdot \frac{\nabla_x T}{\sqrt{T}}, \end{aligned}$$

it remains to find ψ such that:

$$[L_{in}\psi - S_{in}\psi](v_i) = (v_i - u)\cdot \Upsilon.$$

Now, as the right hand side of this equation is an odd function of V_i, and $S_{in} \geqslant 0$, we know from Lemma 4.4 that this equation admits at least one solution (and exactly one if $S_{in} \neq 0$). Moreover, with arguments similar to those developed in ref. 27 (and in ref. 18 for the Lorentz operator), we can show that ψ has to be of the following form: $\bar{\psi}(|v_i - u|)(v_i - u)\cdot\Upsilon$, where the function $\bar{\psi}$ is isotropic with respect to the velocity variable $v_i - u$; this solution also satisfies the orthogonality relation $\int_{I\!R^3} \psi(v) M_{u,T}\, dv = 0$, which gives (56) and (53). ∎

Proof of Lemma 5.5. The aim, in this lemma, is to look for the second order correction $f_2^e = f_0^e\ \phi_2^e$ for the electrons. Referring to the definition of Q_0^{en} given in Lemma 4.1, we first observe that $Q_0^{en}(f_0^e, f_2^n)$ is equal to zero. In a similar way, Lemma 4.2 implies that $Q_0^{ei}(f_0^e, f_1^i) = 0$. Consequently, Eq. (57) reads $L_{en}\phi_2^e = \varphi_2^e$ with L_{en} defined in (20) and

$$\begin{aligned} f_0^e\varphi_2^e &= \partial_t f_0^e + \left[v_e\cdot\nabla_x + F_e\cdot\nabla_{v_e}\right] f_1^e \\ &\quad - Q_0^{en}(f_1^e, f_1^n) - Q_1^{en}(f_1^e, f_0^n) - Q_1^{en}(f_0^e, f_1^n) - 2Q^{ee}(f_1^e, f_0^e) \\ &\quad - Q_0^{ei}(f_1^e, f_0^i) - \mathcal{L}Q^{e,ir}(\phi_1^e, \phi_1^i, \phi_1^n) - Q_2^{en}(f_0^e, f_0^n) - Q_1^{ei}(f_0^e, f_0^i)\ . \end{aligned}$$

Referring to Lemma 4.3, point (ii), we know that if such a solution ϕ_2^e exists, then the right hand side φ_2^e has to satisfy the following orthogonality relation:

$$\int_{I\!R^3} \varphi_2^e(v_e) f_0^e(v_e)\ dv_e = 0. \tag{74}$$

We now compute the different terms involved in this relation. First, mass conservation implies that (see Lemma 4.1.iv and 4.2.ii)

$$\int_{I\!R^3} \Big(Q_0^{en}(f_1^e, f_1^n) + Q_1^{en}(f_1^e, f_0^n) + Q_1^{en}(f_0^e, f_1^n) \\ + \; Q_0^{ei}(f_1^e, f_0^i) + Q_2^{en}(f_0^e, f_0^n) + Q_1^{ei}(f_0^e, f_0^i) \Big) \; dv_e = 0,$$

and the conservative form of Q^{ee} also yields to

$$\int_{I\!R^3} Q^{ee}(f_1^e, f_0^e) \; dv_e = 0.$$

Referring to the definition of $\mathcal{L}Q^{e,ir}$ in Lemma 4.6, simple computations show that:

$$\int_{I\!R^3} \mathcal{L}Q^{e,ir}(\phi_1^e, \phi_1^i, \phi_1^n) \; dv_e = \rho_i \; \mathcal{Y}^{ir},$$

with $\mathcal{Y}^{ir}$ given in (55). Consequently, since we have

$$\int_{I\!R^3} F_e \cdot \nabla_{v_e} f_1^e \; dv_e = 0,$$

the solvability condition (74) reduces to

$$\int_{I\!R^3} \Big(\partial_t f_0^e + v_e \cdot \nabla_x f_1^e \Big) \; dv_e = \rho_i \; \mathcal{Y}^{ir}, \tag{75}$$

which gives:

$$\partial_t \rho_e + \nabla_x \cdot (\rho_e u_1^e) = \rho_i \; \mathcal{Y}^{ir}, \tag{76}$$

where we have set:

$$\rho_e u_1^e = \int_{I\!R^3} v_e f_1^e dv_e.$$

Now, using the expression (52) of f_1^e, we get: $u_1^e = u + u_J$, with u_J defined by (59)–(60). Moreover, ρ_e being connected to ρ_n, ρ_i and T_e by the Saha law (41), we get from the evolution Eqs. (49) and (54) on the densities ρ_n and ρ_i:

$$\partial_t \rho_e = \rho_e \left[-u \left(\frac{\nabla_x \rho_n}{\rho_n} - \frac{\nabla_x \rho_i}{\rho_i} \right) - \mathcal{Y}^{ir} + \partial_t T_e \frac{3T_e + 2\Delta}{2T_e^2} \right], \tag{77}$$

and also

$$\nabla_x \rho_e = \rho_e \left[\frac{\nabla_x \rho_n}{\rho_n} - \frac{\nabla_x \rho_i}{\rho_i} + \nabla_x T_e \frac{3T_e + 2\Delta}{2T_e^2} \right], \tag{78}$$

so that (76) yields to the following equation on T_e:

$$\partial_t T_e + u \cdot \nabla_x T_e + \frac{2T_e^2}{3T_e + 2\Delta} \left(\mathrm{div}(u) + \frac{1}{\rho_e} \mathrm{div}(\rho_e u_J) \right) = \left(1 + \frac{\rho_i}{\rho_e} \right) \frac{2T_e^2}{3T_e + 2\Delta} \mathcal{Y}^{ir}. \tag{79}$$

This concludes the proof. ∎

Proof of Theorem 5.6. This result is a direct consequence of Lemmas 5.1 to 5.5. ∎

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